Functional inequalities for subelliptic heat kernels

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Inégalités fonctionnelles pour des noyaux de la chaleur sous-elliptiques

Dans cette thèse, j'ai étudié le noyau de la chaleur et les inégalités fonctionnelles associées sur trois espaces modèles de la géométrie sous-elliptique. Ces trois espaces sont des groupes de Lie de dimension 3 : le groupe de Heisenberg $\mathbb{H}$, le groupe $\mathbf{SU}(2)$ et le groupe $\mathbf{SL}(2,\mathbb{R})$. Pour chacun de ces groupes, on peut trouver une base de l'algèbre de Lie (X, Y, Z) satisfaisant aux relations :

$$[X,Y] = 2Z, \ [X,Z] = -2\rho Y \text{ et } [Y,Z] = 2\rho X$$

avec $\rho = 1$ pour $\mathbf{SU}(2)$, $\rho = 0$ pour $\mathbb{H}$ et $\rho = -1$ pour $\mathbf{SL}(2,\mathbb{R})$. Comme nous le verrons, ce paramètre ρ a une interprétation en terme de courbure. On munit alors ces trois groupes du sous-laplacien :

$$L = X^2 + Y^2$$

où l'on a identifié les matrices X, Y, Z avec les champs de vecteurs invariants à gauche qu'elles engendrent. Cet opérateur est un opérateur différentiel du second ordre invariant à gauche essentiellement auto-adjoint pour la mesure de Haar du groupe. Il n'est pas elliptique mais, d'après des résultats de Hörmander, il est hypoelliptique. On peut alors aussi construire le semi-groupe de la chaleur associé $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$.

Mes résultats portent tout d'abord sur l'obtention de formules explicites pour les noyaux de la chaleur précédents. Je me suis ensuite intéressé à la généralisation en géométrie sous-elliptique de la notion de courbure de Ricci minorée par une constante. Dans ce sens j'ai introduit un critère de courbure-dimension de Bakry-Emery généralisé qui, sous certaines conditions d'antisymétrie vérifiées sur nos espaces modèles, permet l'obtention d'estimées du type de Li-Yau. Je me suis aussi intéressé à l'établissement et l'étude d'inégalités de sous-commutation entre le gradient et le semi-groupe de la chaleur en redonnant notamment deux nouvelles démonstrations de l'inégalité de H.Q.Li sur $\mathbb{H}$.

Mots clés : semi-groupe de la chaleur, noyau de la chaleur, géométrie sous-elliptique, courbure de Ricci minorée, inégalités fonctionnelles, inégalité de Poincaré, groupe de Heisenberg, estimées de Li-Yau, inégalité de Driver-Melcher, inégalité de H.Q.Li, variétés CR.

Functional inequalities for subellitpic heat kernels

In this thesis, I have studied the heat kernel and the associated functional inequalities on three model spaces in subelliptic geometry. These three spaces are 3-dimensional Lie groups: the Heisenberg group $\mathbb{H}$, $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$. For each of this group, one can find a basis of the Lie algebra (X, Y, Z) which satisfies the relations:

$$[X,Y]=2Z,\ [X,Z]=-2\rho Y \text{ and } [Y,Z]=2\rho X$$

with $\rho=1$ for $\mathbf{SU}(2)$, $\rho=0$ for $\mathbb{H}$ and $\rho=-1$ for $\mathbf{SL}(2,\mathbb{R})$. As we will see it, this parameter ρ has an interpretation in mean of curvature. We then endow these three groups with the sublaplacian:

$$L = X^2 + Y^2$$

where we identify the matrices X, Y, Z with the left-invariant vector fields they engender. This is a left-invariant second order differential opertor. It is self-adjoint with respect to the Haar mesure of the group. It is not elliptic but hypoelliptic from Hörmander's results. One can then construct the associated heat semigroup $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$.

My results first concern the setting of explicit formulas for the above heat kernels. Then I focus on the generalisation in subelliptic geometry of the notion of Ricci curvature bounded below by a constant. In this way, I give a generalized Bakry-Emery curvature-dimension criterion which under some antisymetrical conditions satisfied by our model spaces, implies some Li-Yau type estimates. I also study the setting and the consequences of subcommutation inequalities between the gradient and the semigroup. In particular, I give two new proofs of the H.Q.Li inequality on $\mathbb{H}$.

Keywords: heat semigroup, heat kernel, subelliptic geometry, Ricci curvature bounded below, functional inequalities, Poincaré inequality, Heisenberg group, Li-Yau estimates, Driver-Melcher inequality, H.Q.Li inequality, CR manifolds.

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Chapitre 1

Résumé en français

1.1 Introduction

Dans cette thèse, nous avons étudié le noyau de la chaleur sous-elliptique et les inégalités fonctionnelles associées sur trois espaces différents. Comme nous essaierons de le justifier tout au long de la thèse, les espaces étudiés n'ont pas été choisis au hasard mais correspondent à des espaces modèles de la géométrie sous-elliptique. La notion principale que nous cherchons à comprendre au travers de ces exemples est une généralisation au cadre sous-elliptique de la notion de courbure de Ricci minorée par une constante. Dans le cadre riemannien, la courbure de Ricci décrivant bien le comportement d'un élément de volume le long des géodésiques, cette notion de courbure de Ricci minorée permet d'une part d'obtenir des résultats sur le comportement des boules, par exemple, le théorème de Bishop-Gromov de comparaison de croissance du volume des boules. D'autre part, du fait de la formule de Bochner, cette notion permet d'obtenir aussi de nombreuses inégalités fonctionnelles et notamment des inégalités de Poincaré et de Sobolev logarithmiques locales. Ici, dans cette thèse, nous nous sommes principalement intéressés à ce second point. Plus précisément nous avons cherché à établir de telles inégalités fonctionnelles en géométrie sous-elliptique. L'état d'esprit général de cette thèse consiste à obtenir des résultats sur les trois espaces considérés, tout en espérant que l'on pourra ensuite étendre ces résultats et leurs méthodes à un cadre sous-elliptique plus vaste. La majorité des résultats présentés dans cette thèse sont déjà parus dans les articles publiés [7, 16, 8] et dans la prépublication [25].

Les trois espaces étudiés sont des groupes de Lie de dimension 3, à savoir le groupe de Heisenberg $\mathbb{H}$, le groupe $\mathbf{SU}(2)$ et le groupe $\mathbf{SL}(2,\mathbb{R})$, que l'on munit d'une métrique sous-riemannienne invariante à gauche. A cette métrique est associé de manière canonique un opérateur différentiel du second ordre : le sous-laplacien, que nous noterons L. Cet opérateur est alors hypoelliptique : c'est-à-dire qu'il possède la propriété de régularité suivante : si g est une fonction $\mathcal{C}^\infty$ et f vérifie au sens des distributions $Lf = g$, alors f aussi est $\mathcal{C}^\infty$. De plus, le semi-groupe de la chaleur associé à cet opérateur est bien défini et le noyau de la chaleur correspondant existe, est $\mathcal{C}^\infty$ et est partout strictement positif. Avant de voir en détail cette construction, expliquons un peu le choix des espaces étudiés ainsi que le but de cette étude.

1.2 Présentation des espaces étudiés dans cette thèse

Pour une présentation complète des groupes de Lie $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$ et une description précise de la géométrie sous-elliptique dont on munit ces groupes, on renvoie aux sections 1.1 et 1.3. Disons seulement ici que la métrique dont on munit le groupe $\mathbf{SL}(2,\mathbb{R})$ est celle qui provient de la forme de Killing. Dans cette introduction, ce qui va nous intéresser c'est que l'on peut mettre ces trois espaces dans un cadre commun. Ce cadre commun est celui d'un groupe de Lie de matrices $\mathbb{G}$ dont l'algèbre de Lie $\mathfrak{g}$ admet une base de matrices X, Y, Z vérifiant les relations :

$$[X,Y] = 2Z$$

$$[X,Z] = -2\rho Y$$

$$[Y,Z] = 2\rho X$$

pour un certain paramètre $\rho \in \mathbb{R}$. Comme nous le verrons par la suite, ce paramètre ρ a une interprétation en terme de courbure. Le choix de la constante 2 dans les relations précédentes vient du fait qu'ainsi, pour $\mathbf{SU}(2)$, cas où $\rho = 1$, l'opérateur $X^2 + Y^2 + Z^2$ est exactement l'opérateur de Laplace-Beltrami sur la sphère de dimension 3 et de rayon $\frac{1}{2}$.
Remarquons néanmoins qu'il existe d'autres structures d'algèbres de Lie en dimension 3 non équivalentes à ces trois structures précédentes. Ici, deux algèbres de Lie $\mathfrak{g} = Vect(A,B,C)$ et $\mathfrak{g}' = Vect(A',B',C')$ de dimension 3 sont dites équivalentes s'il existe un isomorphisme d'algèbre de Lie de $\mathfrak{g}$ sur $\mathfrak{g}'$ envoyant $V = Vect(A,B)$ sur $V' = Vect(A',B')$. Une classification de ces structures est donnée dans [93].
L'algèbre de Lie $\mathfrak{g}$ d'un groupe de Lie de matrices $\mathbb{G} \subset M(n,\mathbb{F})$ avec $\mathbb{F} = \mathbb{R}$ ou $\mathbb{C}$ est définie par

$$\mathfrak{g} = \{M \in M(n,\mathbb{F}), \forall t > 0, \exp(tM) \in \mathbb{G}\}$$

et peut être identifiée à l'espace tangent en l'identité du groupe $\mathbb{G}$ vu comme variété différentielle. Les matrices X, Y, Z permettent alors de définir des champs de vecteurs invariants à gauche sur $\mathbb{G}$ que l'on notera encore, avec un petit abus, par les mêmes lettres X, Y, Z. Par exemple, pour une fonction $f \in \mathcal{C}^\infty(\mathbb{G})$ et $g \in \mathbb{G}$, le champ de vecteurs X est défini au point g par

$$X(f)(g) = \frac{d}{dt}_{|t=0} \left(f(g \cdot \exp(tX))\right).$$

Il est bien sûr aussi possible de construire à partir des matrices X, Y, Z des champs de vecteurs invariants à droite sur $\mathbb{G}$, notés $\hat{X}, \hat{Y}, \hat{Z}$ par

$$\hat{X}(f)(g) = \frac{d}{dt}_{|t=0} \left(f(\exp(tX) \cdot g)\right).$$

Maintenant, on peut d'une part munir nos groupes de Lie d'une métrique sous-riemannienne et d'autre part définir le sous-laplacien associé. La métrique sous-riemannienne s'obtient en ne considérant en chaque point de $\mathbb{G}$ que le sous-espace, dit horizontal, $\mathcal{H} = Vect(X,Y)$ de l'espace tangent et en déclarant que la base (X,Y) est orthonormale. On appelle D la distribution horizontale associée, c'est-à-dire l'ensemble des champs de vecteurs U sur $\mathbb{G}$ tels que en tout point $g \in \mathbb{G}$, $U_g \in \mathcal{H}$. La distance sous-riemannienne δ entre 2 points $g, g' \in \mathbb{G}$ est alors définie par :

$$\delta(g,g') = \inf_{\gamma \in \mathcal{A}} \int_0^1 \|\gamma'(t)\| dt$$

où $\mathcal{A}$ désigne l'ensemble des courbes horizontales, c'est-à-dire $\mathcal{C}^1$ par morceaux dont les vecteurs tangents appartiennent au sous-espace $\mathcal{H}$ de l'espace tangent, telles que $\gamma(0) = g$ et $\gamma(1) = g'$. A priori, δ n'est pas forcément une distance, en particulier on n'est pas sûr qu'il existe des courbes horizontales joignant g à g'. Néanmoins dans notre cas, les champs de vecteurs horizontaux ainsi que leurs crochets itérés engendrent toute l'algèbre de Lie (il suffit en fait ici de ne considérer que les champs de vecteurs horizontaux et leurs crochets). Le théorème de Chow (voir [77]) implique alors l'existence de courbes horizontales entre tout couple de points ; δ est alors réellement une distance.

Avant de parler du sous-laplacien, munissons aussi le groupe $\mathbb{G}$ d'une mesure μ. La mesure naturelle que l'on introduit ici est la mesure de Haar invariante à gauche (cette mesure n'est en fait définie qu'à une constante près). Les trois groupes de Lie considérés sont unimodulaires : cela signifie que leur mesure de Haar invariante à gauche est aussi invariante à droite. Cette mesure de Haar correspond en fait à $d\mu = dX \wedge dY \wedge dZ$. L'opérateur qui va nous intéresser dans toute la suite est l'opérateur différentiel du second ordre :

$$L = X^2 + Y^2.$$

Comme les adjoints de X, Y et Z pour la mesure de Haar μ sont : $X^* = -X, Y^* = -Y$ et $Z^* = -Z$, l'opérateur L s'écrit aussi :

$$L = -X^*X - Y^*Y.$$

Cet opérateur sera appelé le sous-laplacien de notre variété sous-riemannienne.

1.3 Construction du semi-groupe associé à l'opérateur L

Dans cette partie, nous allons donner un certain nombre de propriétés satisfaites par l'opérateur L qui vont nous permettre de construire le semi-groupe associé à cet opérateur. Tout d'abord, le sous-laplacien est un opérateur de diffusion. Cela signifie qu'il s'écrit dans une carte sous la forme :

$$L = \sum_{i,j=1}^{3} \sigma_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i(x) \frac{\partial}{\partial x_i}$$

pour des fonctions σ_{ij} et b_i $\mathcal{C}^\infty$ et telles que la matrice $(\sigma_{ij}(x))_{1\leq i,j\leq 3}$ est symétrique et positive. Ici elle n'est pas définie positive, l'opérateur L n'est donc pas elliptique.

L'opérateur L vérifie cependant les propriétés suivantes :

- L est linéaire.
- L est un opérateur local ; c'est-à-dire si $f_1, f_2 \in \mathcal{C}^\infty(\mathbb{G}, \mathbb{R})$ coïncident sur un voisinage de g, alors $L(f_1)(g) = L(f_2)(g)$.
- L vérifie le principe du maximum suivant : si $f \in \mathcal{C}^\infty(\mathbb{G}, \mathbb{R})$ admet un minimum local en g alors $L(f)(g) \geq 0$.

Nous venons de voir que l'opérateur L n'est pas elliptique, néanmoins il possède une propriété de régularité importante : l'hypoellipticité. Cette propriété a été définie par Hörmander [52]. Elle s'énonce ainsi : si g est une fonction $\mathcal{C}^\infty(\mathbb{G}, \mathbb{R})$ et si f vérifie $Lf = g$ au sens des distributions, alors f aussi appartient à $\mathcal{C}^\infty(\mathbb{G}, \mathbb{R})$. Dans un autre de ses articles, fondateur lui aussi, [53], Hörmander a prouvé que si un opérateur L s'écrit comme une somme de carrés de champ de vecteurs sur une variété $\mathcal{M}$, c'est-à-dire si L peut se mettre sous la forme :

$$L = \sum_{i=1}^{d} X_i^2 + X_0$$

pour des champs de vecteurs $(X_i)_{1\le i\le d}$ et X_0 sur $\mathcal{M}$, et si les champs de vecteurs $(X_i)_{1\le i\le d}$ ainsi que leurs crochets itérés engendrent en tout point l'ensemble de l'espace tangent à la variété, alors L est hypoelliptique. Nous appellerons un tel opérateur L un opérateur hypoelliptique de type Hörmander. Les relations de l'algèbre de Lie de $\mathbb{G}$ font ici que cette condition est remplie de manière très simple pour les opérateurs L considérés ici.

Remarque 1.3.1. *Il est possible d'obtenir l'hypoellipticité de L sous des conditions plus faibles utilisant aussi les crochets avec X_0 mais ces hypothèses ne suffisent pas pour obtenir la stricte positivité du noyau de la chaleur. Ici n'étudiant que des sous-laplaciens, nous nous restreindrons aux opérateurs décrits ci-dessus.*

Il est aussi possible d'associer une distance d_L à l'opérateur L, cette distance est définie par :

$$d_L(g, g') = \sup_{f\in\mathcal{C}} f(g) - f(g') \tag{1.3.1}$$

où $\mathcal{C}$ désigne l'ensemble des fonctions $f \in \mathcal{C}^\infty(\mathbb{G}, \mathbb{R})$ telles que $\Gamma(f, f) \le 1$ avec

$$\Gamma(f, f) = \frac{1}{2}\left(Lf^2 - 2fLf\right).$$

Ici,

$$\Gamma(f, f) = (Xf)^2 + (Yf)^2.$$

Cette distance coïncide en fait exactement avec la distance sous-riemannienne définie précédemment (voir par exemple [59]).
Maintenant nous allons voir comment construire le semi-groupe de la chaleur associé à l'opérateur L. Tout d'abord, il est facile de voir que l'opérateur L est symétrique sur l'ensemble des fonctions $\mathcal{C}_c^\infty(\mathbb{G}, \mathbb{R})$ par rapport à la mesure de Haar μ : si $f_1, f_2 \in \mathcal{C}_c^\infty(\mathbb{G}, \mathbb{R})$, alors

$$\int_{\mathbb{G}} f_1\, Lf_2 d\mu = \int_G Lf_1\, f_2 d\mu.$$

Ceci provient simplement du fait que pour la mesure de Haar μ : $X^* = -X$ et $Y^* = -Y$.
Nous rappelons qu'un opérateur symétrique sur $\mathcal{C}_c^\infty(\mathbb{G}, \mathbb{R})$ par rapport à une mesure borélienne est dit essentiellement auto-adjoint s'il existe une unique extension sur un domaine dense de $L^2_\mu(\mathbb{G}, \mathbb{R})$ auto-adjointe. Remarquons aussi que pour un opérateur symétrique et positif, il existe une extension minimale canonique en un opérateur auto-adjoint. Cette construction est appelée extension de Friedrichs.
La proposition suivante va nous permettre de montrer que les opérateurs L considérés ici sont essentiellement auto-adjoints. Pour une référence à propos de cette proposition, on peut consulter [83] pour le cas elliptique et [84] pour le cas précis qui nous intéresse.

Proposition 1.3.2. *Soit L un opérateur de diffusion hypoelliptique de type Hörmander sur une variété $\mathcal{M}$ à coefficient $\mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$. S'il existe une suite de fonctions $h_n \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$ telle que $0 \le h_n \le 1$, $h_n \nearrow 1$ et $\|\Gamma(h_n,h_n)\|_\infty \to 0$ quand $n \to \infty$, alors l'opérateur L est essentiellement auto-adjoint sur $\mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$.*
De plus la métrique associée à L est complète.

Il est facile de voir que les opérateurs L sur $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$ vérifient bien les hypothèses de la Proposition 1.3.2. De plus, lorsque nous essaierons d'étendre nos résultats à des opérateurs L sous-elliptiques plus généraux, nous supposerons toujours que les hypothèses de la proposition 1.3.2 sont vérifiées.

On peut alors définir le semi-groupe $(P_t)_{t\ge0} = (e^{tL})_{t\ge0}$ grâce au théorème spectral pour un opérateur auto-adjoint et on obtient un opérateur de contraction fortement continu dans $L^2_\mu(\mathcal{M},\mathbb{R})$ de générateur L.
Le semi-groupe $(P_t)_{t\ge0}$ est aussi un semi-groupe sous-markovien (voir par exemple le chapitre 1 de [45]) , c'est-à-dire :

$$\text{si } u \in L^2_\mu(\mathbb{G},\mathbb{R}) \text{ et } 0 \le u \le 1, \text{ alors } 0 \le P_t u \le 1 \ \mu-\text{presque sûrement.}$$

Par le théorème d'interpolation de Riesz-Thorin, on peut montrer que le semi-groupe $(P_t)_{t\ge0}$ est défini de manière unique dans $L^p_\mu(\mathcal{M},\mathbb{R})$, $1 \le p \le \infty$.
Enfin, les résultats de Hörmander impliquent aussi que la solution fondamentale du semi-groupe, que l'on appellera noyau de la chaleur par la suite, appartient à l'espace $\mathcal{C}^\infty(\mathcal{M},\mathbb{R})$ et est strictement positive sur tout $\mathcal{M}$.

1.4 Les espaces étudiés dans cette thèse sont des espaces modèles

Les trois espaces étudiés sont ici pensés comme des espaces modèles de la géométrie sous-elliptique en dimension 3. Dans ce sens notre conviction est qu'ils devraient jouer les mêmes rôles en géométrie sous-elliptique que l'espace euclidien, la sphère et l'espace hyperbolique en géométrie riemannienne. Le petit bémol venant du fait que le groupe $\mathbf{SL}(2,\mathbb{R})$ homéomorphe en tant que variété différentielle à $\mathbb{R}^2 \times S^1$ n'est pas simplement connexe, il serait sans doute intéressant d'étudier le revêtement universel $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ de $\mathbf{SL}(2,\mathbb{R})$. L'espace euclidien, la sphère et l'espace hyperbolique sont des espaces modèles de la géométrie riemannienne car ils sont simplement connexes et à courbure sectionnelle constante. De nombreux théorèmes en géométrie riemannienne sont des théorèmes de comparaison avec ces espaces modèles. Par exemple, le théorème de Bishop-Gromov (voir [46]) assure que si une variété riemannienne $\mathcal{M}$ de dimension n possède une courbure de Ricci minorée par $(n-1)kId$, alors le ratio

$$\frac{v_p(r)}{V(r)}$$

est décroissant, où $v_p(r)$ désigne le volume de la boule centrée en $p \in \mathcal{M}$ et de rayon r et $V(r)$ le volume de la boule de même rayon r dans l'espace modèle de courbure sectionnelle constante égale à k et de dimension n. Cet espace modèle est soit la sphère de rayon $\frac{1}{\sqrt{k}}$ si $k > 0$, soit l'espace euclidien si $k = 0$, soit l'espace hyperbolique de courbure k si $k < 0$. De plus, cela conduit aussi à la comparaison $v_P(R) \le V(R)$ pour tout $R > 0$.

Nous n'avons pas réellement cherché à developper cet aspect dans notre cadre sous-elliptique. Les seuls résultats connus dans cette direction portent en fait sur la propriété de contraction de la mesure. Cette propriété de contraction de la mesure est liée à une comparaison du Jacobien le long des géodésiques issues d'un point (voir [86] et [79] pour cette propriété dans le cadre riemannien). Juillet [61] a ainsi montré que le groupe de Heisenberg satisfait à la propriété $MCP(0,5)$ et très récemment Agrachev et Lee [2] ont defini une propriété de contraction de la mesure $MCP(\rho;2,3)$ où la paire $(2,3)$ designe la dimension de la distribution et ont montré que cette propriété était satisfaite dans le cas de certaines fibrations englobant les cas de $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$. Néanmoins, leur comparaison du Jacobien ne permet pas de donner des théorèmes de comparaison de volume des boules (voir [31] pour le calcul du Jacobien dans un cadre général ou encore [61] ou la partie 7.3.2 de cette thèse pour le cas du groupe de Heisenberg).

L'aspect de la géométrie riemannienne que nous avons vraiment cherché à étendre est celui des inégalités fonctionnelles reposant sur le formalisme Γ_2. Dans le cas riemannien, cette méthode repose sur la formule de Bochner qui lie le Γ_2 d'un opérateur avec la courbure de Ricci associée à l'opérateur. Le Γ_2 d'un opérateur L est donné par les formules :

$$\Gamma(f,g) = \frac{1}{2}\left(L(fg) - fLg - gLf\right)$$

et enfin :

$$\Gamma_2(f,g) = \frac{1}{2}\left(L(\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\right).$$

Dans le cas où L est l'opérateur de Laplace-Beltrami sur une variété riemannienne de dimension n, la formule de Bochner s'écrit

$$\Gamma_2(f,f) = \|Hessf\|_2^2 + Ric(\nabla f, \nabla f).$$

Ainsi, si sur la variété, le tenseur de Ricci est minoré par ρId, en utilisant aussi l'inégalité de Cauchy-Schwarz $\|Hessf\|_2^2 \geq \frac{1}{n}(Lf)^2$, on obtient l'inégalité suivante, vérifiée pour toute fonction $f \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$:

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{n}(Lf)^2.$$

Cette inégalité est exactement le critère de courbure dimension de Bakry-Emery $CD(\rho,n)$. Si l'on ne prend pas en compte le terme dimensionnel $\frac{1}{n}(Lf)^2$, on obtient le critère plus faible infini-dimensionnel $CD(\rho,\infty)$:

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f).$$

Ces critères s'appliquent aussi pour des opérateurs de la forme : $\Delta + \nabla V.\nabla$. Par exemple, l'espace $\mathbb{R}^n$ muni de la mesure gaussienne est l'exemple typique d'un espace vérifiant le critère $CD(1,\infty)$. Maintenant, pour obtenir par exemple des vitesses de convergence vers la mesure invariante ou bien des propriétés de régularisation du semi-groupe ou encore des estimées du noyau de la chaleur, deux différentes approches ont été particulièrement développées. La première est l'obtention à partir du ctitère $CD(\rho,n)$ d'estimées de Li-Yau sur le gradient du logarithme d'une solution positive du noyau de la chaleur. Ces estimées permettent alors d'obtenir par intégration le long des géodésiques des inégalités de Harnack puis des estimées du noyau de la chaleur, ainsi que le théorème de compacité de Myers et des inégalités isopérimétriques pour la mesure invariante.

La seconde est basée sur une équivalence entre le critère $CD(\rho,\infty)$, la sous-commutation entre le gradient et le semi-groupe et une foultitude d'inégalités fonctionnelles locales (c'est-à-dire pour la mesure $P_t(.)(x)$) incluant des inégalités de Poincaré et Poincaré inverse, de Sobolev Logarithmique et Sobolev Logarithmique inverse, des inégalités isopérimétriques de type Cheeger et de type Bobkov...

Dans le cadre sous-elliptique considéré ici, le Γ_2 de l'opérateur est bien défini et il vaut :

$$\Gamma_2(f,f) =(X^2f)^2 + (Y^2f)^2 + \frac{1}{2}\left((XY+YX)f\right)^2 + 2(Zf)^2 \tag{1.4.2}$$

$$+ 4\rho\Gamma(f,f) - 4(Xf)(YZf) + 4(Yf)(XZf). \tag{1.4.3}$$

Intuitivement, les termes croisés $-4(Xf)(YZf) + 4(Yf)(XZf)$ empêchent le critère $CD(\rho,\infty)$ d'être verifié. Ceci peut être montré de manière rigoureuse. A noter aussi que la généralisation de la notion courbure de Ricci minorée à un espace métrique donnée indépendemment par Lott et Villani [73] et par Sturm [85, 86] n'est pas non plus satisfaite sur le groupe de Heisenberg (voir [61]). Cette notion de courbure de Ricci minorée est basée sur des propriétés de convexités des géodésiques dans l'espace des probabilités sur l'espace métrique mesuré. Pour plus de détails sur ces points, on renvoie à la partie 2.2 de la thèse. Il ne semble donc pas y avoir de bonne notion de courbure de Ricci minorée dans ce cadre sous-elliptique. La motivation première de cette thèse a donc consisté à essayer d'étendre ces deux types de résultat précédents dans notre cadre sous-elliptique.

Le plan de la thèse est le suivant. Dans le chapitre 2, nous décrivons de manière précise les groupes $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$ ainsi que les structures sous-elliptiques dont nous les munissons. Nous expliquons aussi pourquoi les notions existantes de courbure de Ricci minorée par une constante ne s'appliquent pas. Enfin nous introduisons des coordonnées cylindriques bien adaptées à nos espaces sous-elliptiques que nous utiliserons dans toute la suite et nous établirons quelques propriétés de symétrie pour le noyau de la chaleur. Le chapitre 3 concernant les variétés CR est inclus dans cette thèse pour deux raisons. La première c'est de justifier que les espaces que nous étudions sont bien des espaces modèles, la seconde est d'introduire une classe d'espaces (les variétés CR dont la torsion pseudo-Hermitienne de Tanaka-Webster s'annule) pour laquelle les techniques du chapitre 5 s'appliquent. Le chapitre 4 est consacré à une chose importante dont nous avons peu parlé jusqu'ici, à savoir l'obtention d'expressions explicites pour les noyaux de la chaleur et leurs conséquences. Dans le chapitre 5 nous établissons des inégalités sous-elliptiques du type de Li-Yau. La méthode utilisée est basée sur le formalisme Γ_2 et généralise celle utilisée par Bakry et Ledoux dans [12]. Dans le cadre riemannien, les inégalités de Li-Yau relient des bornes inférieures sur la courbure de Ricci à des bornes du gradient du noyau de la chaleur. Les estimées obtenues ici, tout comme dans le cadre riemannien permettent d'obtenir des inégalités de Harnack, des théorèmes de compacités de Myers ainsi que des inégalités isopérimétriques. Le chapitre 6 est lui consacré à une inégalité fonctionnelle particulière : l'inégalité de Poincaré locale inverse. L'intérêt principal de ce chapitre est que cette inégalité se démontre de manière élémentaire et que l'on obtient la constante optimale pour cette inégalité. Enfin, dans le chapitre 7, nous nous intéressons à des inégalités de sous-commutation entre le gradient et le semi-groupe. Dans un premier temps nous décrivons les conséquences d'une telle inégalité dans un cadre général. Les inégalités fonctionnelles qui en découlent sont en fait presque les mêmes que celles obtenues sous les critères $CD(\rho,\infty)$ à savoir des inégalités de Poincaré, Sobolev logarithmique, isopérimétriques de type Cheeger et Bobkov. Dans un deuxième temps, nous essayons d'obtenir ces inégalités de sous-commutation pour nos espaces modèles.

Dans la suite de ce résumé en français, nous décrivons plus en détails les résultats obtenus dans cette thèse.

1.5 Expressions explicites des noyaux de la chaleur

Le premier point important de notre travail porte donc sur l'obtention de formules explicites du noyau de la chaleur sous-elliptique et sur l'étude des propriétés qui en découlent. Ce travail est décrit dans le chapire 4 de cette thèse. Sur le groupe de Heisenberg, l'expression du noyau de la chaleur est bien connue, elle est donnée par la formule dite de Gaveau qui a été montrée par Gaveau [48] et Hulanicki [57] et qui, en fait, était déjà contenue dans les travaux de Lévy sur l'aire balayée par un mouvement brownien dans $\mathbb{R}^2$ [69]. Pour pouvoir obtenir des formules de representation du noyau de la chaleur sur les deux espaces restants, la première chose que nous avons faite a été de choisir des coordonnées bien adaptées à notre problème. Nous avons introduits les coordonnées cylindriques suivantes :

$$(r, \theta, z) \to \exp\left(r\cos\theta X + r\sin\theta Y\right)\exp(zZ).$$

Dans ces coordonnées il est alors possible d'obtenir par des calculs simples mais un peu pénibles les expressions des champs de vecteurs puis de l'opérateur L.
On obtient les expressions suivantes,
– pour $\mathbb{H}$:

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + 2\frac{\partial^2}{\partial z\partial\theta}, \tag{1.5.4}$$

– pour $\mathbf{SU}(2)$:

$$L = \frac{\partial^2}{\partial r^2} + 2\ \mathrm{cotan}\ 2r\frac{\partial}{\partial r} + \tan^2 r\frac{\partial^2}{\partial z^2} + \left(\frac{2}{\sin 2r}\right)^2\frac{\partial^2}{\partial\theta^2} + 2\tan r\left(\frac{2}{\sin 2r}\right)\frac{\partial^2}{\partial z\partial\theta}, \tag{1.5.5}$$

– pour $\mathbf{SL}(2,\mathbb{R})$:

$$L = \frac{\partial^2}{\partial r^2} + 2\coth 2r\frac{\partial}{\partial r} + \tanh^2 r\frac{\partial^2}{\partial z^2} + \left(\frac{2}{\sinh 2r}\right)^2\frac{\partial^2}{\partial\theta^2} + 2\tanh r\left(\frac{2}{\sinh 2r}\right)\frac{\partial^2}{\partial\theta\partial z}. \tag{1.5.6}$$

Le noyau de la chaleur sur le groupe de Heisenberg est donné par la proposition suivante :

Proposition 1.5.1. *Par rapport à la mesure de Lebesgue $rdrd\theta dz$, le noyau de la chaleur associé au semigroupe $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ sur $\mathbb{H}$ s'écrit*

$$h_t(r,z) = \frac{1}{16\pi^2 t^2}\int_{-\infty}^{+\infty} e^{\frac{i\lambda z}{2t}} e^{-\frac{r^2}{4t}\lambda\coth\lambda}\frac{\lambda}{\sinh\lambda}d\lambda. \tag{1.5.7}$$

A l'aide de cette expression, en étudiant la fonction holomorphe en la variable y sous l'intégrale, Gaveau [48] a pu obtenir les asymptotiques du noyau de la chaleur en temps court, puis dans un autre papier plus récent avec Beals et Greiner [19], les estimées optimales de ce noyau (voir aussi [56] et [70]). Dans cette thèse nous donnons une démonstration de la formule de Gaveau utilisant la transformée de Fourier sphérique sur le groupe Heisenberg due à Faraut [43].

Pour le groupe $\mathbf{SU}(2)$, Nous avons réussi à obtenir par un argument très simple une décomposition spectrale du noyau de la chaleur sous-elliptique :

Proposition 1.5.2. *Par rapport à la mesure* $d\mu = \frac{\sin 2r}{2} dr d\theta dz$*, le noyau de la chaleur associé au semigroupe* $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ *sur* $\mathbf{SU}(2)$ *s'écrit, pour* $t > 0$*,* $r \in [0, \pi/2)$*,* $z \in [-\pi, \pi]$*,*

$$p_t(r, z) = \frac{1}{2\pi^2} \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} (2k + \mid n \mid +1) e^{-(4k(k+|n|+1)+2|n|)t} e^{inz} (\cos r)^{|n|} P_k^{0,|n|}(\cos 2r),$$

où

$$P_k^{0,|n|}(x) = \frac{(-1)^k}{2^k k! (1+x)^{|n|}} \frac{d^k}{dx^k} \left((1+x)^{|n|} (1-x^2)^k \right)$$

est un polynôme de Jacobi.

Cette expression s'obtient en prenant la série de Fourier en z : $\sum_{n\in\mathbb{Z}} e^{inz} \Phi_n(t, r)$ d'une fonction solution de l'equation de la chaleur. Ceci fait apparaître l'oscillateur harmonique sur la sphère de dimension 2 et de rayon 1 :

$$\frac{\partial \Phi_n}{\partial t} = \frac{\partial^2 \Phi_n}{\partial r^2} + 2 \text{ cotan } 2r \frac{\partial \Phi_n}{\partial r} - n^2 \tan^2 r \Phi_n.$$

Puis en faisant un bon changement de fonction nous arrivons à nous ramener à l'équation de Jacobi :

$$\frac{\partial g_n}{\partial t}(x) = 4\mathcal{G}_n(g_n)(x)$$

où

$$\mathcal{G}_n = (1-x^2) \frac{\partial^2}{\partial x^2} + (|n| - (2 + |n|)x) \frac{\partial}{\partial x}$$

dont les fonctions propres sont bien connues : les polynômes de Jacobi. Nous pouvons alors décomposer le noyau de la chaleur comme une somme de ces polynômes de Jacobi. Les constantes peuvent alors être calculées grâce aux propriétés d'orthogonalité des polynômes de Jacobi et à la condition initiale que doit satisfaire le noyau de la chaleur. Il est à noté que cette expression avait déjà été obtenue par Bauer [18] à l'aide de la théorie des representations de $\mathbf{SU}(2)$.
Nous n'avons malheureusement pas réussi à obtenir par cette méthode une décomposition spectrale du noyau de la chaleur sur $\mathbf{SL}(2, \mathbb{R})$.

Par contre, nous avons aussi réussi à obtenir des représentations intégrales nouvelles des noyaux de la chaleur sur $\mathbf{SU}(2)$ et sur $\mathbf{SL}(2, \mathbb{R})$. Ces représentations intégrales sont en fait très liées aux submersions décrites dans la section 2.3 de la thèse. Pour le groupe $\mathbf{SU}(2)$ qui s'identifie à la sphère unité S^3, cela nous donne le lien très fort entre le sous-laplacien L et l'opérateur de Laplace-Beltrami canonique Δ_{S^3} sur S^3. En effet ces deux opérateurs sont reliés par :

$$\Delta_{S^3} = L + Z^2$$

avec de plus la relation

$$[L, Z] = 0.$$

Ceci va nous permettre d'écrire :

$$e^{tL} = e^{-tZ^2} e^{t\Delta_{S^3}}.$$

Remarquons aussi que, avec notre choix de coordonnées, le champ de vecteurs Z a une expression très simple puisqu'il s'écrit

$$Z = \frac{\partial}{\partial z}.$$

Pour pouvoir obtenir par cette méthode une représentation du noyau de la chaleur sous-elliptique, il nous faut d'abord déterminer l'expression du noyau de la chaleur elliptique associé à Δ_{S^3} dans nos coordonnées.

Lemme 1.5.3. *Si $f \in \mathcal{C}_c^\infty(\mathbf{SU}(2), \mathbb{R})$, alors pour $t \geq 0$,*

$$(e^{t\Delta} f)(0) = \frac{1}{4\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\pi}^{\pi} q_t(\cos r \cos z) f(r, \theta, z) \sin 2r dr d\theta dz$$

avec

$$q_t(x) = \sum_{m=0}^{+\infty} (m+1) e^{-m(m+2)t} U_m(x), \quad x \in [-1, 1]. \tag{1.5.8}$$

Nous sommes alors maintenant à même d'obtenir l'expression suivante :

Proposition 1.5.4. *Par rapport à la mesure $d\mu = \frac{\sin 2r}{2} dr d\theta dz$, le noyau de la chaleur associé au semigroupe $(P_t)_{t\geq 0} = (e^{tL})_{t \geq 0}$ sur $\mathbf{SU}(2)$ s'écrit aussi, pour $t > 0$, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$,*

$$p_t(r, z) = \frac{1}{2\pi^2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+iz)^2}{4t}} q_t(\cos r \cosh y) dy.$$

Pour le groupe $\mathbf{SL}(2, \mathbb{R})$, ce coup-ci c'est avec l'opérateur de Casimir $\square$ que le sous-laplacien a un lien étroit. On a en effet les relations suivantes :

$$L = \square + Z^2$$

et de plus

$$[L, Z] = 0.$$

L'opérateur de Casimir n'est pas un opérateur elliptique mais un opérateur hyperbolique. Etant plus familier avec les opérateurs elliptiques, nous avons préferré travailler avec le noyau de la chaleur sur l'espace hyperbolique de dimension 3 et avons réussi à obtenir l'expression intégrale suivante :

Proposition 1.5.5. *Par rapport à la mesure $d\mu = \frac{\sinh 2r}{2} dr d\theta dz$, le noyau de la chaleur associé au semigroupe $(P_t)_{t\geq 0} = (e^{tL})_{t \geq 0}$ sur $\mathbf{SL}(2, \mathbb{R})$ s'écrit, pour $t > 0, r > 0, z \in [-\pi, \pi]$,*

$$p_t(r, z) \quad = \quad \frac{1}{4\pi} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} s_t(\cosh r \cosh y) dy$$

avec

$$s_t(\cosh r) = \frac{e^{-t}}{\sqrt{4\pi}\, t^{3/2}} \left(\frac{r}{\sinh r} \right) e^{-\frac{r^2}{4t}}. \tag{1.5.9}$$

1.6 Asymptotiques en temps court du noyau de la chaleur

L'intérêt principal de ces expressions intégrales réside dans le fait que, le comportement du noyau de la chaleur canonique elliptique sur S^3 ou H^3 étant bien connu, il est alors possible d'obtenir des informations sur le comportement des noyaux sous-elliptiques, par exemple des asymptotiques en temps petit. La méthode que nous utilisons ici pour obtenir ces asymptotiques est la même que celle utilisée par Gaveau dans son papier [48]. Le principe en est le suivant : on cherche à estimer l'intégrale d'une fonction en la variable y, holomorphe dans un certain domaine contenant la droite réelle, de la forme

$$\int_{y=-\infty}^{\infty} \exp\left(-\frac{f(r,z,y)}{4t}\right) V(r,y)dy.$$

On cherche alors les points critiques de la fonction f, on trouve dans la bande $|Im(y)| < \pi$, un unique point critique. Ce point critique est imaginaire pur et sera donc noté $i\theta(r,z)$. Il se trouve qu'en ce point, la dérivée seconde de la fonction est un nombre réel strictement positif. On peut alors changer le contour d'intégration dans le plan complexe et donner un équivalent de l'intégrale par la méthode de Laplace, $V(0)$ étant non nul,

$$\int_{\mathbb{R}} e^{-\frac{f(y)}{4t}} V(y)dy \sim \sqrt{\frac{2\pi}{f''(0)}}\sqrt{t}e^{-\frac{f(0)}{4t}} V(0). \tag{1.6.10}$$

Les résultats que nous avons obtenus sont les suivants :

Proposition 1.6.1.

– *Pour* $\mathbb{H}$,

$$h_t(0,z) = \frac{1}{8t^2} \frac{e^{-\frac{\pi z}{2t}}}{\left(1+e^{-\frac{\pi z}{2t}}\right)^2},$$

$$h_t(r,0) \sim \frac{1}{2(4\pi t)^{\frac{3}{2}}} \frac{\sqrt{3}}{r} e^{-\frac{r^2}{4t}}$$

et pour $r > 0, z \neq 0$,

$$h_t(r,z) \sim \frac{1}{(4\pi t)^{3/2}} \frac{\sin\theta(r,z)}{r} \sqrt{\frac{\sin\theta(r,z)}{\theta(r,z)\cos\theta(r,z) - \sin\theta(r,z)}} e^{-\left(\frac{r^2\theta(r,z)\cot\theta(r,z) - 2z\theta(r,z)}{4t}\right)}$$

où $\theta(r,z)$ *est l'unique solution dans* $[-\pi,\pi]$ *de l'équation*

$$\left(\frac{\theta(r,z)}{\sin^2\theta(r,z)} - \text{cotan}\theta(r,z)\right) r^2 = 2z.$$

– *Pour* $\mathbf{SU}(2)$,

$$p_t(0,z) = \frac{e^t}{8t^2} \frac{e^{-\frac{2\pi z - z^2}{4t}}}{\left(1+e^{-\frac{\pi z}{2t}}\right)^2} \left(1 + O(e^{-\frac{C}{t}})\right),$$

$$p_t(r,0) \sim \frac{1}{2(4\pi t)^{\frac{3}{2}}} \frac{r}{\sin r} \sqrt{\frac{1}{1 - r\text{cotan} r}} e^{-\frac{r^2}{4t}}$$

et pour $r > 0, z \neq 0$,

$$p_t(r,z) \sim \frac{1}{2(4\pi t)^{\frac{3}{2}}} \frac{1}{\sin r} \frac{\arccos u(r,z)}{\sqrt{1 - \frac{u(r,z)\arccos u(r,z)}{\sqrt{1-u^2(r,z)}}}} e^{-\frac{(\theta(r,z)-z)^2 \tan^2 r}{4t \sin^2 \theta(r,z)}}$$

avec $u(r,z) = \cos r \cos\theta(r,z)$ *et où* $\theta(r,z)$ *est l'unique solution dans* $[-\pi,\pi]$ *de l'équation*

$$\theta(r,z) - z = \cos r \sin\theta(r,z) \frac{\arccos(\cos\theta(r,z)\cos r)}{\sqrt{1-\cos^2 r \cos^2\theta(r,z)}}.$$

– *Pour* $\mathbf{SL}(2,\mathbb{R})$,

$$p_t(0,z) = \frac{e^{-t}}{8t^2} \frac{e^{-\frac{2\pi z + z^2}{4t}}}{\left(1+e^{-\frac{\pi z}{2t}}\right)^2},$$

$$p_t(r,0) \sim \frac{1}{2} \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{r}{\sinh r} \sqrt{\frac{1}{r \coth r - 1}} e^{-\frac{r^2}{4t}}$$

et pour $r > 0, z \neq 0$,

$$p_t(r,z) \sim \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{1}{\sinh r} \frac{\operatorname{arccosh} u(r,z)}{\sqrt{\frac{u(r,z)\operatorname{arccosh} u(r,z)}{\sqrt{u^2(r,z)-1}} - 1}} e^{-\frac{(\theta(r,z)-z)^2 \tanh^2 r}{4t \sin^2 \theta(r,z)}}$$

avec $u(r,z) = \cosh r \cos\theta(r,z)$ *et où* $\theta(r,z)$ *est l'unique solution dans* $[-\pi,\pi]$ *de l'équation*

$$\theta(r,z) - z = \cosh r \sin\theta(r,z) \frac{\operatorname{arch}(\cosh r \cos\theta(r,z))}{\sqrt{\cosh^2 r \cos^2\theta(r,z) - 1}}.$$

Pour les points de la forme $(0,z)$, le résultat est en fait obtenu par un calcul de la valeur exacte en ce point à l'aide de la formule des résidus.

Ces asymptotiques combinées aux résultats de Léandre [65, 64] donnent alors les expressions variationnelles suivantes pour la distance sous-riemannienne.

Proposition 1.6.2.
– *Dans le cas de* $\mathbb{H}$, *on a*
  – *pour* $z \in \mathbb{R}$,
$$d^2_{\mathbb{H}}(0,z) = 2\pi \mid z \mid .$$
  – *pour* $r > 0$,
$$d^2_{\mathbb{H}}(r,0) = r^2.$$
  – *pour* $r > 0,\ z \in \mathbb{R}$
$$d^2_{\mathbb{H}}(r,z) = r^2\theta(r,z)\cot\theta(r,z) - 2z\theta(r,z).$$
– *Dans le cas de* $\mathbf{SU}(2)$, *on a*
  – *Pour* $z \in [-\pi,\pi]$,
$$d^2_{\mathbf{SU}(2)}(0,z) = 2\pi \mid z \mid - z^2.$$

– *Pour* $r > 0$,

$$d^2_{\mathbf{SU}(2)}(r,0) = r^2.$$

– *Pour* $z \in [-\pi, \pi]$, $r \in \left(0, \frac{\pi}{2}\right)$,

$$d^2_{\mathbf{SU}(2)}(r,z) = \frac{(\theta(r,z) - z)^2 \tan^2 r}{\sin^2 \theta(r,z)}.$$

– *Dans le cas de* $\mathbf{SL}(2,\mathbb{R})$, *on a*
  – *For* $z \in [-\pi, \pi]$,

$$d^2_{\mathbf{SL}(2,\mathbb{R})}(0,z) = 2\pi \mid z \mid + z^2.$$

  – *For* $r > 0$,

$$d^2_{\mathbf{SL}(2,\mathbb{R})}(r,0) = r^2.$$

  – *For* $z \in [-\pi, \pi]$, $r > 0$,

$$d^2_{\mathbf{SL}(2,\mathbb{R})}(r,z) = \frac{(\theta(r,z) - z)^2 \tanh^2 r}{\sin^2 \theta(r,z)}.$$

Enfin, ces expressions explicites du noyau de la chaleur permettent de montrer un résultat de convergence pour les diffusions. Ce résultat est plus fort que le résultat de Mitchell [76] où la convergence a lieu seulement au niveau de la métrique.

Proposition 1.6.3. *Pour* $\mathbf{SU}(2)$ *et* $\mathbf{SL}(2,\mathbb{R})$, *uniformément sur les sous-ensembles compacts* $\mathbb{R}_{\geq 0} \times \mathbb{R}$,

$$\lim_{t \to 0} t^2 p_t(\sqrt{t}r, tz) = h_1(r,z).$$

1.7 Estimées sous-elliptiques de type de Li-Yau

Avant de présenter les estimées sous-elliptiques de type Li-Yau que nous avons obtenues, commençons par faire un petit rappel sur l'inégalité parabolique de Li-Yau dans le cadre riemannien. Cette inégalité constitue un outil puissant pour obtenir des estimées sur les noyaux de la chaleur, sujet qui a connu une intense activité au cours des trentes dernières années (voir par exemple [72, 36]). En géométrie riemannienne, elle relie des bornes du gradient du noyau de la chaleur à une borne inférieure de la courbure de Ricci. Plus précisement, dans sa fome la plus simple, si $\mathcal{M}$ est une variété riemannienne de dimension n et de courbure de Ricci positive, et si f est une solution positive de l'équation de la chaleur :

$$\partial_t f = \Delta f,$$

où Δ est l'opérateur de Laplace-Beltrami sur $\mathcal{M}$, alors avec $u = \ln f$, l'inégalité de Li-Yau s'écrit :

$$\partial_t u \geq |\nabla u|^2 - \frac{n}{2t}.$$

Beaucoup de généralisations de cette inégalité ont été développées, toutes incluant des bornes inférieures du tenseur de courbure de Ricci. Elle a notamment été étendue au cas d'un opérateur elliptique général L vérifiant le critère de courbure-dimension $CD(\rho, n)$ qui comme on l'a vu précédemment, généralise la notion de borne inférieure de la courbure de Ricci (voir [13, 12]).

La méthode classique de Li et Yau [72] consiste à appliquer le principe du maximum à une quantité bien choisie. Une méthode différente utilisant le formalisme Γ_2 a été développée par Bakry et Ledoux [12]. Leur principe est de considérer une solution positive de l'équation de la chaleur $\partial_t f = Lf$ et d'introduire la fonction auxiliaire

$$\Phi(s) = P_s(f(t-s)\Gamma(\ln f(t-s), \ln f(t-s)))$$

définie pour $0 < s < t$.
L'inégalité $CD(\rho, n)$ implique alors une inégalité différentielle du type

$$\Phi'(s) \geq (A\Phi(s) + B)^2 + C,$$

où A, B, C sont des expressions constantes en t mais qui dépendent de la fonction f. L'inégalité parabolique de Li-Yau découle alors de cette inégalité différentielle.

Ici dans le chapitre 5, nous développons un peu plus cette méthode dans un cadre sous-elliptique en essayant d'obtenir des inégalités différentielles pour des fonctions plus compliquées du type :

$$P_s(f(t-s)(a(s)\Gamma(\ln f(t-s), \ln f(t-s)) + b(s)(Z \ln f(t-s))^2)).$$

Dans cette partie, notre cadre de travail est le suivant. Nous considérons une variété $\mathcal{M}$ de dimension $2n+1$ et des champs de vecteurs $(X_i)_{1 \leq i \leq 2n}$ et Z sur la variété satisfaisant aux relations

$$[X_i, X_j] = \sum_{l=1}^{2n} \omega_{ij}^l X_l + \gamma_{ij} Z \tag{1.7.11}$$

et

$$[X_i, Z] = \sum_{l=1}^{2n} \delta_i^l X_l \tag{1.7.12}$$

où $1 \leq i, j \leq 2n$, $\omega_{ij}^l, \gamma_{ij}$ et δ_i^l sont des fonctions $\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ telles que $\omega_{ij}^l = -\omega_{ji}^l$, $\gamma_{ij} = -\gamma_{ji}$ et avec la relation particulièrement importante :

$$\delta_i^l = -\delta_l^i. \tag{1.7.13}$$

Bien évidemment, notre cadre de travail englobe les cas de $\mathbb{H}$, $\mathbf{SU}(2)$ et $\mathbf{SL}(2, \mathbb{R})$. D'un point de vue géométrique, cela correspond à se donner une variété CR dont la torsion pseudo hermitienne de la connection de Tanaka Webster est nulle (voir le chapitre 3 de la thèse). L'opérateur L que l'on considère est alors :

$$L = \sum_{i=1}^{2n} X_i^2 + \sum_{i,k=1}^{2n} \omega_{ik}^i X_k.$$

Cet opérateur est le sous-laplacien canonique sur une variété CR et coïncide bien sûr avec le sous-laplacien décrit précédemment sur $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2, \mathbb{R})$. On considère alors aussi P_t le semi-groupe associé à l'opérateur L, bien défini d'après la théorie décrite précédemment. L'intérêt du cadre précédent provient en fait des deux relations suivantes :

$$[L, Z] = 0 \tag{1.7.14}$$

et pour toute fonction $f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$

$$\sum_{i=1}^{2n} X_i(f)[X_i, Z](f) = 0. \tag{1.7.15}$$

Ces relations nous donnent alors le lemme suivant :

Lemme 1.7.1. *Soit f une fonction positive $f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$, et posons pour $0 \leq s \leq t$,*

$$\Phi_1(s) = P_s((P_{t-s}f)\Gamma(\ln P_{t-s}f))(x)$$

et

$$\Phi_2(s) = P_s((P_{t-s}f)(Z \ln P_{t-s}f)^2)(x).$$

Alors les dérivées de Φ_1 et Φ_2 s'obtiennent par :

$$\Phi_1'(s) = 2P_s((P_{t-s}f)\Gamma_2(\ln P_{t-s}f))(x)$$

et

$$\Phi_2'(s) = 2P_s((P_{t-s}f)\Gamma(Z \ln P_{t-s}f))(x).$$

Maintenant nous allons nous placer aussi dans le cas où le critère suivant de courbure-dimension est satisfait, pour toute fonction $g \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$ et tout $\lambda > 0$:

$$\Gamma_2(g) \geq \frac{1}{2}(Lg)^2 + 2(Zg)^2 + \left(4\rho - \frac{2}{\lambda}\right)\Gamma(g) - 2\lambda\Gamma(Zg). \tag{1.7.16}$$

Ce critère de courbure est une généralisation au cadre sous-elliptique du critère de courbure $CD(\rho, n)$ de Bakry-Emery. De plus, grâce à la formule du Γ_2 (1.4.2) et à une inégalité de Cauchy-Schwarz, on voit facilement que ce critère est satisfait avec le ρ correspondant pour les espaces $\mathbb{H}, \mathbf{SU}(2)$ and $\mathbf{SL}(2, \mathbb{R})$.
Sous ces hypothèses, nous obtenons alors le système différentiel suivant :

Proposition 1.7.2. *Pour toute fonction dérivable, décroissante, positive $b : [0, t] \to \mathbb{R}$, on a*

$$\left(-\frac{b'}{4}\Phi_1 + b\Phi_2\right)'(s) \geq -\frac{b'(s)}{4}\left(\left(\frac{b''(s)}{b'(s)} + 2\frac{b'(s)}{b(s)} + 8\rho\right)LP_tf(x) - \frac{1}{4}\left(\frac{b''(s)}{b'(s)} + 2\frac{b'(s)}{b(s)} + 8\rho\right)^2 P_tf(x)\right).$$

C'est à partir de ce système différentiel et de bons choix de fonctions b que nous obtiendrons toutes les estimées suivantes. Par exemple, en prenant une fonction b décroissante positive telle que $b(t) = b'(t) = 0$, alors

$$\Gamma(\ln P_tf) + \frac{-4b(0)}{b'(0)}Z(\ln P_tf)^2 \leq A(t)\frac{LP_tf}{P_tf} + B(t) \tag{1.7.17}$$

avec

$$A(t) = \int_0^t \frac{b'(s)}{b'(0)}(-\gamma(s))ds,$$

$$B(t) = \frac{1}{4}\int_0^t \frac{b'(s)}{b'(0)}\gamma(s)^2 ds$$

et

$$\gamma(s) = \frac{b''}{b'} + 2\frac{b'}{b} + 8\rho.$$

Le choix $b(s) = (t-s)^\alpha$ pour $\alpha > 2$ pour $\rho \geq 0$ donne le résultat suivant :

Proposition 1.7.3. *Pour tout $\alpha > 2$, toute fonction f positive et tout $t > 0$,*

$$\Gamma(\ln P_t f) + \frac{t}{\alpha}(Z \ln P_t f)^2 \leq \left(\frac{3\alpha - 1}{\alpha - 1} - \frac{2\rho t}{\alpha} \right) \frac{\mathcal{L} P_t f}{P_t f} + \frac{\rho^2 t}{\alpha} - \frac{\rho(3\alpha - 1)}{\alpha - 1} + \frac{(3\alpha - 1)^2}{\alpha - 2} \frac{1}{t} \quad (1.7.18)$$

Dans le cas $\rho \geq 0$, cela se simplifie en :

Corollaire 1.7.4. *Sur $\mathbb{H}$ et $\mathbf{SU}(2)$, il existe des constantes A, B et C telles que :*

$$\Gamma(\ln P_t f) + Bt(Z \ln P_t f)^2 \leq A\partial_t \ln(P_t f) + \frac{B}{t}.$$

Quand $\rho > 0$, nous pouvons obtenir une décroissance exponentielle en choisissant la fonction

$$b(s) = \left(e^{-\frac{8\rho s}{3\alpha}} - e^{-\frac{8\rho t}{3\alpha}} \right)^\alpha, \quad \alpha > 2.$$

Corollaire 1.7.5. *Pour tout $\rho > 0$ et $\alpha > 2$, pour toute fonction f, $x \in \mathbf{G}$ et $t > 0$,*

$$\Gamma(\ln P_t f) + \frac{3}{2}\left(1 - e^{-\frac{8\rho t}{3\alpha}}\right)(Z \ln P_t f)^2 \leq 3\,(3\alpha - 1)\,\frac{\alpha}{\alpha - 1} e^{-\frac{8\rho t}{3\alpha}} \frac{L P_t f}{P_t f} + 6\rho \frac{(3\alpha - 1)^2}{\alpha(\alpha - 2)} \frac{e^{-\frac{16\rho t}{3\alpha}}}{1 - e^{-\frac{8\rho t}{3\alpha}}}. \quad (1.7.19)$$

Dans le cas $\rho > 0$, par une étude précise du système différentiel précédent, nous pouvons aussi retrouver la compacité de l'espace ainsi qu'une borne explicite du diamètre (non optimale cependant) : pour tout $\alpha > 2$,

$$diam(\mathcal{M}) \leq \frac{3}{\sqrt{2}} \sqrt{\frac{(\alpha - 1)(3\alpha - 1)}{(\alpha - 2)}} \frac{1}{\sqrt{\rho}}.$$

En intégrant les estimées de Li-Yau le long des géodésiques, nous obtenons les inégalités de Harnack suivantes :

Proposition 1.7.6. – *Sur $\mathbb{H}$ et $\mathbf{SU}(2)$, il existe deux constantes A_1 et A_2 telles que pour tout $0 < t_1 < t_2$:*

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{A_1} \exp\left(A_2 \frac{\delta(g_1, g_2)^2}{t_2 - t_1}\right) \quad (1.7.20)$$

– *Sur $\mathbf{SL}(2, \mathbb{R})$, il existe deux constantes B_1 et B_2 telles que pour tout $0 < t_1 < t_2 \leq 1$ $g_1, g_2 \in \mathbf{SL}(2, \mathbb{R})$*

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{B_1} \exp\left(B_2 \frac{\delta_{\mathbf{SL}(2,\mathbb{R})}(g_1, g_2)^2}{t_2 - t_1}\right) \quad (1.7.21)$$

et il existe deux constantes $\tilde{B}_1$ et $\tilde{B}_2$ telles que pour tout $2 < t_1 < t_2$ et $g_1, g_2 \in \mathbf{SL}(2, \mathbb{R})$

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \exp\left(\tilde{B}_1(t_2 - t_1)\right) \exp\left(\tilde{B}_2 \frac{\delta_{\mathbf{SL}(2,\mathbb{R})}(g_1, g_2)^2}{t_2 - t_1}\right). \quad (1.7.22)$$

En utilisant des méthodes développées par Varopoulos [91] et reprises par Ledoux [67], nous pouvons aussi montrer à partir des estimées de Li-Yau des inégalités isopérimétriques. Ainsi pour le groupe de Heisenberg on retrouve le résultat suivant :

Proposition 1.7.7. *Il existe une constante C tel que pour tout A ensemble de Cacciopoli de $\mathbb{H}$*

$$\mu(A)^{\frac{3}{4}} \leq CP(A)$$

où $P(A)$ est le périmètre de A.

Remarquons que 4 est la dimension homogène du groupe de $\mathbb{H}$. Pour le groupe $\mathbf{SL}(2,\mathbb{R})$, nous n'obtenons malheureusement par cette méthode que ce résultat pour des ensembles réguliers de volume suffisemment petits alors que le résultat est en fait valable pour tout les ensembles réguliers (voir [31]). Dans le cas de $\mathbf{SU}(2)$, qui est un ensemble compact, nous obtenons en renormalisant la mesure en une mesure de probabilité :

Proposition 1.7.8. *Il existe une constante C telle que pour tout A ensemble de Cacciopoli de $\mathbf{SU}(2)$*

$$\mu(A)(1-\mu(A)) \leq C\frac{1}{\sqrt{\rho}}P(A).$$

Cette inégalité est en fait équivalent à une inégalité de Poincaré L^1.
Comme dernière conséquence, nous obtenons une borne uniforme du gradient du logarithme du noyau de la chaleur en fonction de la distance. Cette borne n'est ici démontrée que pour les espaces modèles $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$. Une borne de ce même type a été obtenue sur les variétés riemanniennes à courbure de Ricci minorée par Engoulatov [42]. Elle s'écrit ici :

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq C\left(\frac{d(r,z)}{t}+\frac{1}{\sqrt{t}}\right)$$

pour une constante C et où $d(r,z)$ désigne la distance sous-riemannienne de l'identité au point de coordonnées (r,θ,z). Cette distance, tout comme le noyau de la chaleur issu de l'identité, ne dépend pas de la variable θ. Encore une fois, pour $\mathbf{SL}(2,\mathbb{R})$ en temps grand nous obtenons une estimée plus faible :

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq C\left(\frac{d(r,z)}{t}+1\right).$$

Enfin, nous étudions l'inégalité différentielle 1.7.2 d'une manière différente : nous cherchons ce coup-ci des fonctions a et b telles que $(a\Phi_1+b\Phi_2)'(s) \geq 0$ de telle sorte que l'on puisse comparer les fonctions Φ_1 et Φ_2 au temps 0 et au temps t. L'analyse effectuée ici ne tient pas compte du terme dimensionnel $\frac{1}{2}(Lg)^2$ et donc peut être réalisée aussi pour les fonctions

$$\Psi_1(s) = P_s\left(\Gamma(P_{t-s}f)\right)$$

et

$$\Psi_2(s) = P_s\left(Z(P_{t-s}f)^2\right).$$

Les résultats obtenus sont alors les suivants :

Proposition 1.7.9. *Soit L un opérateur de diffusion qui satisfait aux hypothèses de la proposition 1.7.2 avec $\rho > 0$. Si f est une fonction $\mathcal{C}^\infty(\mathcal{M},\mathbb{R})$, alors*

$$\Gamma(P_tf)+\frac{3}{2\rho}Z(P_tf)^2 \leq e^{-\frac{8\rho t}{3}}P_t\left(\Gamma(f)+\frac{3}{2\rho}Z(f)^2\right).$$

Proposition 1.7.10. *Soit L un opérateur de diffusion qui satisfait aux hypothèses de la proposition 1.7.2 avec $\rho \leq 0$. Si f est une fonction $\mathcal{C}^\infty(\mathcal{M},\mathbb{R})$, alors*

$$\Gamma(P_tf) + Z(P_tf)^2 \leq e^{|8(\rho-1)|t} P_t \left(\Gamma(f) + Z(f)^2\right).$$

Ces résultats peuvent aussi s'exprimer comme une inégalité de sous-commutation entre le gradient modifié

$$\tilde{\Gamma}(f) = \Gamma(f) + cZ(f)^2, \text{ pour } C > 0$$

et le semi-groupe P_t.
En définissant

$$\tilde{\Gamma_2}(f) = \frac{1}{2}\left(L\tilde{\Gamma}(f,f) - 2\tilde{\Gamma}(f,Lf)\right),$$

nous obtenons alors une généralisation d'un théorème classique présenté dans cette thèse (théorème 1.9.1) :

Proposition 1.7.11. *Soit $k \in \mathbb{R}$, les affirmations suivantes sont équivalentes :*
- $\forall f \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$, $\tilde{\Gamma_2}(f) \geq k\tilde{\Gamma}(f)$
- $\forall f \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$, $\tilde{\Gamma}(P_tf) \leq e^{-2kt}P_t(\tilde{\Gamma}(f))$
- $\forall f \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$, $(P_tf)\tilde{\Gamma}(\ln P_tf) \leq e^{-2kt}P_t(f\tilde{\Gamma}(\ln f))$.

Dans ce cas, contrairement au théorème 1.9.1, nous ne savons pas si l'inégalité

$$\sqrt{\tilde{\Gamma}(P_tf)} \leq e^{-kt}P_t(\sqrt{\tilde{\Gamma}(f)})$$

est satisfaite ou non. Néanmoins, on peut quand même obtenir des inégalités locales de Poincaré et Sobolev logarithmique faisant intervenir le gradient modifié.

Proposition 1.7.12. *Si l'une des assertions du théorème précédent est satisfaite pour un certain $k \in \mathbb{R}$, alors pour tout $f \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$,*

$$P_t(f^2) - P_t(f)^2 \leq \frac{1-e^{-2kt}}{k} P_t(\tilde{\Gamma}(f))$$

et

$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \leq \frac{1-e^{-2kt}}{k}\, P_t\left(\frac{\tilde{\Gamma}(f)}{f}\right).$$

1.8 Inégalité de Poincaré inverse

Nous décrivons ici les résultats du chapitre 6. Dans ce chapitre, nous montrons qu'il est possible d'obtenir de manière simple et directe des inégalités du type Poincaré inverse. L'intérêt principal est qu'ici nous obtenons la constante optimale dans ces inégalités. Remarquons aussi que dans le chapitre 7 nous obtenons une inégalité de Poincaré inverse par la sous-commutation entre le gradient et le semi-groupe. La lettre G désigne ici l'un des groupes $\mathbb{H}, \mathbf{SU}(2)$ ou $\mathbf{SL}(2,\mathbb{R})$. Nous décrivons d'abord l'inégalité où intervient la variance pour la mesure réversible.

Proposition 1.8.1. *Soit $f \in \mathcal{C}_c^\infty(\mathbb{G},\mathbb{R})$ à support compact. Pour $t>0$ et $g \in \mathbb{G}$,*

$$\Gamma(P_t f, P_t f)(g) \leq A(t) \left(\int_{\mathbb{G}} f^2 d\mu - \left(\int_{\mathbb{G}} f d\mu \right)^2 \right)$$

avec

$$A(t) = -\frac{1}{4}\frac{d}{dt}\int_{\mathbb{G}} p_t^2 d\mu.$$

Nous obtenons ensuite l'inégalité de Poincaré inverse proprement dite où intervient cette fois la variance pour la mesure locale $P_t()(x)$.

Proposition 1.8.2. *Soit $f \in \mathcal{C}_c^\infty(\mathbb{G},\mathbb{R})$, pour $t>0$ et $g \in \mathbb{G}$,*

$$\Gamma(P_t f, P_t f)(g) \leq C(t) \left(P_t f^2(g) - (P_t f)^2(g) \right)$$

avec

$$C(t) = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{G}} p_t \ln p_t d\mu.$$

Les démonstrations de ces inégalités utilisent fortement la structure de groupe de Lie invariant à gauche, et tout comme pour la démonstration de Driver-Melcher (voir section 1.11 pour plus de détails) :

$$X(P_t f)(0) = \hat{X} P_t(f)(0) = P_t(\hat{X} f)(0)$$

puis une intégration par parties :

$$P_t(\hat{X} f)(0) = -\int_{\mathbb{G}} \hat{X} p_t \, f d\mu$$

Nous appliquons enfin une inégalité de Hölder bien choisie. Nous utilisons alors des propriétés de symétrie pour bien trouver la constante optimale.
Dans ce chapitre nous décrivons aussi les comportements des constantes optimales sur nos trois espaces modèles. La constante, ou simplement son comportement, $A(t)$ peut s'obtenir en remarquant que

$$\int_{\mathbb{G}} p_t^2 d\mu = p_{2t}(0)$$

et en utilisant les expressions explicites du noyau de la chaleur.
Le comportement de la constante $C(t)$ est un peu plus complexe à obtenir.

Proposition 1.8.3. *Sur $\mathbb{H}$, pour tout $t>0$,*

$$C(t) = \frac{1}{t}.$$

Quand t tend vers 0, sur $\mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$, nous avons aussi :

$$C(t) \sim_{t\to 0} \frac{1}{t}. \tag{1.8.23}$$

Quand t tend vers ∞, nous avons le comportement suivant :
– sur $\mathbf{SU}(2)$, $C(t) \sim_{t\to+\infty} 4e^{-4t}$.

Nous n'avons malheureusement pas réussi à décrire totalement le comportement de la constante $C(t)$ en temps grand sur $\mathbf{SL}(2,\mathbb{R})$.

1.9 Inégalité de sous-commutation entre le gradient et le semi-groupe

Le point de départ du chapitre 7 de cette thèse est le suivant. Dans notre cadre sous-elliptique, le critère de courbure de Bakry-Emery $CD(\rho,\infty)$ n'est pas satisfait et par conséquent le théorème ci-dessous n'est pas vérifié :

Théorème 1.9.1. *Soit L un opérateur de diffusion vérifiant les hypothèses de la proposition 1.3.2 et $\rho \in \mathbb{R}$, les affirmations suivantes sont équivalentes :*

1. *le critère $CD(\rho,\infty)$ est satisfait*
2. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ \sqrt{\Gamma(P_t f)} \leq e^{-\rho t} P_t(\sqrt{\Gamma f})$
3. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ \Gamma(P_t f) \leq e^{-2\rho t} P_t(\Gamma f)$.

Néanmoins il reste possible que des inégalités de commutation plus générales entre gradient et semi-groupe soient satisfaites :

$$\Gamma(P_t f) \leq C_2(t) P_t(\Gamma f) \tag{1.9.24}$$

et

$$\sqrt{\Gamma(P_t f)} \leq C_1(t) P_t(\sqrt{\Gamma f}). \tag{1.9.25}$$

Bien sûr, si tel est le cas, les fonctions $C_1(t)$ et $C_2(t)$ optimales ne peuvent pas présenter n'importe quel comportement en $t = 0$ puisqu'alors $C_1(0) = C_2(0) = 1$. Nous verrons qu'en fait les égalités que nous obtenons ne sont pas continues en $t = 0$.
Les premiers résultats de ce type sont dus à Driver et Melcher dans [39] où ils prouvent que l'inégalité (1.9.24) est satisfaite sur le groupe de Heisenberg avec $C_2(t)$ égale à une constante $C_2 \geq 2$ pour $t > 0$. Ensuite H.Q. Li [70] a prouvé que l'inégalité plus forte (1.9.25) était aussi satisfaite sur le groupe de Heisenberg avec $C_1(t)$ égale à une constante $C_1 \geq \sqrt{2}$ pour $t > 0$.
Il est aussi important de noter qu'une conjecture de Coulhon-Duong [33] énonce des liens entre les inégalités (1.9.24) et (1.9.25) et le caractère borné de la transformée de Riesz.

1.10 Les conséquences de l'inégalité de sous-commutation

Le but de ce chapitre est double, il s'agit d'une part de comprendre en termes d'inégalités fonctionnelles les conséquences des inégalités (1.9.24) et (1.9.25) et d'autre part d'établir ces inégalités dans le cas de $\mathbb{H}, \mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$. Pour le premier point, nous verrons que presque toutes les inégalités habituellement obtenues sous le critère $CD(\rho,\infty)$ peuvent en fait être obtenues sous l'hypothèse plus faible (1.9.25). Nous décrivons ici tous les résultats obtenus dans un même théorème.

Théorème 1.10.1. *Soit L un opérateur de diffusion satisfaisant aux hypothèses de la proposition 1.3.2 et soit P_t le semi-groupe associé. Supposons que l'inégalité (1.9.25) est vérifiée pour toute fonction $f \in \mathcal{C}_c^\infty(\mathcal{M},\mathbb{R})$ avec une certaine fonction $C(t)$, alors les inégalités suivantes sont aussi satisfaites :*

1. *l'inégalité de Poincaré, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$:*

$$P_t(f^2) - P_t(f)^2 \leq 2\left(\int_0^t C_1(u)^2 du\right) P_t(\Gamma(f))$$

2. *l'inégalité de Beckner-Latała-Oleszkiewicz,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, $f > 0$ *et* $p \in (1,2]$ *:*
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \leq p \left(\int_0^t C_1(u)^2 du \right) P_t(f^{p-2}\Gamma(f))$$

3. *l'inégalité de Sobolev logarithmique,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, $f \geq 0$ *:*
$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \leq \left(\int_0^t C_1(u)^2 du \right) P_t\left(\frac{\Gamma(f)}{f} \right)$$

4. *l'inégalité de Poincaré inverse,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$ *:*
$$P_t(f^2) - P_t(f)^2 \geq 2 \left(\int_0^t \frac{1}{C_1(u)^2} du \right) \Gamma(P_t f)$$

5. *l'inégalité de Beckner-Latała-Oleszkiewicz inverse,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, $f > 0$ *et* $p \in (1,2]$ *:*
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \geq p \left(\int_0^t \frac{1}{C_1(u)^2} du \right) (P_t f)^{p-2}\Gamma(P_t f)$$

6. *l'inégalité de Sobolev logarithmique,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, $f > 0$ *:*
$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \geq \left(\int_0^t \frac{1}{C_1(u)^2} du \right) \frac{\Gamma(P_t f)}{P_t f}$$

7. *l'inégalité isopérimetrique de Cheeger,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, $\forall \mathbf{x} \in \mathcal{M}$ *:*
$$P_t(|f - P_t(f)(\mathbf{x})|)(\mathbf{x}) \leq 2R(t)P_t(\sqrt{\Gamma}(f))(\mathbf{x})$$

8. *une première inégalité isopérimetrique de Bobkov,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$ *:*
$$\mathcal{I}(P_t f) - P_t(\mathcal{I} f) \leq R(t)P_t(\sqrt{\Gamma} f)$$

9. *une seconde inégalité isopérimetrique de Bobkov,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$ *:*
$$\mathcal{I}(P_t f) \leq P_t\left(\sqrt{\mathcal{I}(f)^2 + R(t)^2\Gamma(f)} \right)$$

10. *une inégalité isopérimetrique de Bobkov inverse,* $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$ *:*
$$\mathcal{I}(P_t f) - P_t(\mathcal{I} f) \geq r(t)\sqrt{\Gamma P_t f}$$

avec
$$R(t) = \int_0^t C_1(s) \left(\int_0^s \frac{2}{C_1(u)^2} du \right)^{-\frac{1}{2}} ds,$$
$$r(t) = \int_0^t \left(2\int_0^s C_1(u)^2 du \right)^{-1/2} \frac{1}{C_1(s)} ds$$

et $\mathcal{I} : [0,1] \to [0, (2\pi)^{-1/2}]$ *la fonction isopérimétrique gaussienne définie par* $\mathcal{I} = (F_\gamma)' \circ (F_\gamma)^{-1}$ *où*
$$F_\gamma(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{1}{2}u^2} du.$$

La principale différence avec le cas elliptique est qu'ici nous n'avons qu'une implication dans ce théorème. Dans le cadre elliptique où le critère $CD(\rho,\infty)$ est vérifié, toutes les inégalités précédentes sont en fait des égalités pour $t=0$ et on peut comparer les dérivées en $t=0$ et obtenir l'implication inverse.
Il est aussi possible d'obtenir une autre inégalité isoperimétrique de type Cheeger pour les fonctions s'annulant sur une boule.

Théorème 1.10.2. *Soit $t\geq 0$, $\mathbf{x}\in M$, et B une boule de M, il existe une constante $C_{B,t,\mathbf{x}}>1$ telle que pour toute fonction $f\in\mathcal{C}_c^\infty(M,\mathbb{R})$ s'annulant sur B,*

$$|P_t(f)(\mathbf{x})|\leq C_{B,t,\mathbf{x}}\,P_t(\sqrt{\Gamma(f)})(\mathbf{x}). \tag{1.10.26}$$

Ce résultat peut en fait être étendu à tout borélien de mesure non nulle.
Remarquons aussi que la seconde inégalité isopérimétrique de Bobkov, qui est celle qui se tensorise bien, n'est pas déduite directement de la sous-commutation (1.9.25) mais se déduit de la première inégalité isopérimétrique de Bobkov par un argument de Barthe et Maurey [14] qui dit que les deux inégalités sont en fait équivalentes. Dans le cas elliptique $CD(\rho,\infty)$, il existe aussi une seconde inégalité isopérimétrique de Bobkov inverse que, pour le moment, nous ne savons pas déduire dans le seul cadre de sous-commutation (1.9.25).
Remarquons enfin que dans le cas où juste (1.9.24) est satisfaite, nous ne pouvons déduire que l'inégalité de Poincaré et l'inégalité de Poincaré inverse.

1.11 L'obtention des inégalités de sous-commutation

Nos résultats les plus intéressants ont été obtenus dans le cas du groupe de Heisenberg. Nous avons notamment établi deux nouvelles démonstrations plus simples de l'inégalité de H.Q. Li, l'une basée sur une inégalité de type Cheeger et l'autre sur une quasi-commutation du semi-groupe avec un gradient complexe. Tout d'abord, avant de donner les grandes idées de ces deux preuves, redonnons une démonstration simple de l'inégalité de Driver-Melcher.
Par invariance à gauche, on peut simplement travailler en l'identité, on remarque alors que les champs de vecteurs X et $\hat{X}$ coïncident en ce point et que de plus $\hat{X}$ et P_t commutent, on a donc :

$$X(P_tf)(0)=\hat{X}(P_tf)(0)=P_t(\hat{X}f)(0).$$

Il reste alors à écrire $\hat{X}=X+2yZ$ et $2Z=XY-YX$, faire des intégrations par parties pour ce dernier terme et enfin utiliser une inégalité de Cauchy-Schwarz. Il faut ensuite faire de même pour le terme $Y(P_tf)(0)$.

Venons-en maintenant à la démonstration de l'inégalité de H.Q. Li via l'inégalité de Cheeger. Le début de la démonstration est le même que celui de celle de l'inégalité de Driver-Melcher, simplement pour la suite, nous faisons un découpage à l'aide de fonctions lisses, nous utilisons les estimées précises du gradient du noyau de la chaleur et, pour faire le recollement, nous utilisons une inégalité de Cheeger du type de celle du théorème 1.10.2 que nous devons ici montrer à la main. La preuve à la main de l'inégalité de Cheeger se fait en utilisant la structure des géodésiques du groupe de Heisenberg ainsi que les estimées optimales du noyau de la chaleur. Le point clé est en effet l'obtention de l'inégalité :

$$\int_t^{2\pi|u|}A(u,s)h(u,s)ds\leq C(t_0)A(u,t)h(u,t),\ \forall t\geq t_0>0 \tag{1.11.27}$$

où (u,s) parcourt les géodésiques du groupe de Heisenberg et $A(u,s)$ dénote la mesure de Lebesgue dans ces coordonnées géodésiques. En fait, nous avons besoin d'une version un peu plus forte de cette inégalité de Cheeger dont la preuve utilise aussi une inégalité de Poincaré L^1 sur les boules.

La seconde démonstration de l'inégalité de H.Q. Li est basée sur la commutation :

$$(X+iY)L = (L-4iZ)(X+iY),$$

qui conduit à la commutation formelle :

$$(X+iY)P_t = e^{t(L-4iZ)}(X+iY). \tag{1.11.28}$$

Cette commutation est seulement formelle puisque le semi-groupe complexe $e^{t(L-4iZ)}$ n'est pas bien défini globalement. Cependant en étudiant les propriétés d'holomorphie en la variable z du noyau de la chaleur $h_t(r,z)$, nous établissons que ce semi-groupe est bien défini contre les gradients de fonctions et de plus nous obtenons une représentation intégrale de ce semi-groupe. Le noyau de cette représentation n'est cependant pas unique. En effet, formellement le noyau que l'on a envie de considérer est

$$h_t(r,\tilde{z}) \text{ avec } \tilde{z} = z + 4it.$$

Seulement ce noyau admet un pôle double en $\tilde{z} = i\left(2t + \frac{r^2}{4}\right)$. Par contre, vu que nous ne considérons ce noyau que contre des gradients de fonctions $(X+iY)f$, nous pouvons y retrancher toute fonction telle que son gradient complexe est nul. Ainsi nous pouvons considérer le noyau h_t^* :

$$h_t^*(r,z) = h_t(r,z) - \frac{1}{8\pi^2\left(t+iz+\frac{r^2}{4}\right)^2}.$$

Ce noyau ne possède pas de pôles. Nous obtenons donc la représentation suivante :

Proposition 1.11.1. *Si $f : \mathbb{H} \to \mathbb{R}$ est une fonction lisse à support compact, alors*

$$(X+iY)P_tf(0) = \int_{\mathbb{H}} h_t^*(r, z+4it)(X+iY)f(r,\theta,z)rdrd\theta dz, \quad t>0.$$

Maintenant si le rapport

$$\frac{|h_t^*(r,z+4it)|}{h_t(r,z)}$$

était borné, nous aurions terminé et aurions obtenu l'inégalité de H.Q. Li. Malheureusement ce n'est pas le cas (ça l'est bien sûr sur tout sous-ensemble compact), nous pouvons néanmoins considérer le noyau $h_t(r,z+4it)$ contre des fonctions lisses à support compact f dont le support ne contient pas le pôle. Le fait que sur un compact ne contenant pas le pôle, le rapport

$$\frac{|h_t(r,z+4it)|}{h_t(r,z)}$$

est borné peut se démontrer de la même manière que les estimées optimales de $h_t(r,z)$ obtenues dans [19]. Pour conclure et faire le recollement entre les deux estimées précédentes, nous utilisons aussi l'inégalité de Poincaré L^1 sur les boules dont nous avons déjà parlée précédemment.

Nous pouvons en fait montrer qu'il n'existe pas de noyaux tels que le rapport précédent soit borné partout (voir le lemme 7.3.13 et ses conséquences).

Nous nous sommes aussi intéressés aux groupes de Heisenberg $\mathbb{H}_n$ de dimension supérieure. L'inégalité de H.Q. Li reste encore valable et a été démontrée par Elredge [41] en étendant la méthode basée sur l'inégalité de Cheeger. Ici nous ne ferons pas une nouvelle démonstration complète de cette inégalité mais nous verrons que la méthode basée sur la commutation complexe fonctionne encore et, chose surprenante, que lorsque $n \geq 3$, l'opérateur qui apparaît possède un noyau bien défini (sans pôles).

Penchons nous maintenant sur les deux espaces restants $\mathbf{SU}(2)$ et $\mathbf{SL}(2,\mathbb{R})$. Le problème pour adapter les preuves précédentes est que l'on ne dispose pas des estimations optimales du noyau de la chaleur dans ces deux cas là (voir section 7.4.4 pour une discussion sur ce sujet). Néanmoins, nous pouvons remarquer que la quasi-commutation complexe peut se faire et se comporte de manière similaire à celle sur le groupe de Heisenberg. En effet, avec le paramètre ρ décrit précédemment, nous avons la commutation :

$$(X+iY)L = (L - 4iZ + 4\rho)(X+iY) \tag{1.11.29}$$

qui conduit à la commutation formelle

$$(X+iY)P_t = e^{t(L-4iZ+4\rho)}(X+iY). \tag{1.11.30}$$

Les propriétés analytiques du noyau $e^{4\rho t}p_t(r, z+4it)$ sont similaires à celle pour le groupe de Heisenberg, à savoir l'existence d'un pôle double que nous pouvons faire diaparaître contre les gradients complexes de fonctions en retranchant une quantité bien choisie. Nous obtenons alors les propositions suivantes :

Proposition 1.11.2. *Si $f : \mathbf{SU}(2) \to \mathbb{R}$ est une fonction lisse à support compact, alors*

$$(X+iY)P_t f(0) = e^{4t}\int_{\mathbb{H}} p_t^*(r, z+4it)(X+iY)f(r,\theta,z)d\mu, \quad t>0$$

avec

$$p_t^*(r,z) = p_t(r,z) - \frac{1}{2\pi^2}\frac{1}{\left(1-\cos r e^{-iz-2t}\right)^2}.$$

Proposition 1.11.3. *Si $f : \mathbf{SL}(2,\mathbb{R}) \to \mathbb{R}$ est une fonction lisse à support compact, alors*

$$(X+iY)P_t f(0) = e^{-4t}\int_{\mathbb{H}} p_t^*(r, z+4it)(X+iY)f(r,\theta,z)d\mu, \quad t>0$$

avec

$$p_t^*(r,z) = p_t(r,z) - \frac{1}{2\pi^2}\frac{1}{\left(1-\frac{1}{\cosh r}e^{-iz-2t}\right)^2}.$$

Sur $\mathbf{SU}(2)$, en utilisant la décomposition spectrale du noyau de la chaleur, nous montrons que, pour $t \geq t_0 > 0$:

$$|p_t^*(r, z+4it)| \leq C(t_0)e^{-6t}$$

et obtenons le corollaire suivant :

Corollaire 1.11.4. *Pour tout $t_0 > 0$, il existe une constante $A > 0$ telle que pour toute fonction $f : \mathbf{SU}(2) \to \mathbb{R}$ lisse à support compact,*

$$\sqrt{\Gamma(P_t f, P_t f)(0)} \leq A e^{-2t} P_t \sqrt{\Gamma(f,f)}(0), \quad t \geq t_0.$$

Il est alors possible d'obtenir cette sous-commuation en temps court aussi. Malheureusement, nous ne l'obtenons ici que pour une puissance strictement plus grande que la racine carrée.

Proposition 1.11.5. *Soit $p > 1$. Il existe une constante $A_p > 0$ telle que pour toute fonction $f : \mathbf{SU}(2) \to \mathbb{R}$ et $g \in \mathbf{SU}(2)$*

$$\sqrt{\Gamma(P_t f, P_t f)(g)} \leq A_p \left(P_t \Gamma(f,f)^{\frac{p}{2}}(g) \right)^{\frac{1}{p}}, t \in (0,1).$$

Ceci se démontre à l'aide d'intégrations par parties similaires à celles de la preuve de l'inégalité de Driver-Melcher sur Heisenberg, des expressions explicites des champs de vecteurs sur $\mathbf{SU}(2)$ et d'une inégalité de Hölder.

Chapter 2

Presentation of the model spaces

2.1 The model spaces

The goal behind this thesis is to understand how, in a subriemanian context, the "curvature" appears in the heat kernel and in the associated functional inequalities. More precisely, we are interested in a notion of Ricci curvature bounded from below. The idea was then to begin to study the simplest examples of this kind of geometry, that is why we restrict to the dimension 3 and that we work only on some "models" spaces to begin this study. As we shall see it later, for these spaces there is no good notion of such curvature for the moment.
In this thesis, we will therefore study three subelliptic structures over some 3 dimensional manifolds. These structures should play the role of constant curvature subelliptic manifolds in dimension 3. One structure shall be the model space of the positively curved subelliptic 3-dimensional manifold, another the model space of a flat subelliptic manifold and the last one the model space of a negatively curved subelliptic manifold. The 3 structures are in fact carried by some Lie groups $\mathbb{G}$ equipped with a left invariant metric.
Here the Lie groups are in fact some subgroups of $GL(n, F)$ with $F = \mathbb{R}$ or $\mathbb{C}$ and $n = 2$ or 3. The Lie algebra $\mathfrak{g}$ of such a group $\mathbb{G}$ is then

$$\mathfrak{g} = \{M \in M(n, F), \exp(tM) \in \mathbb{G}, \forall t > 0\}.$$

This Lie algebra $\mathfrak{g}$ can be identified with the tangent space of $\mathbb{G}$ in the identity and the tangent space at a point g of $\mathbb{G}$ can be identified with $g.\mathfrak{g}$. Moreover a matrix $M \in \mathfrak{g}$ induces a left-invariant vector field on $\mathbb{G}$. A left-invariant vector field is a vector field V such that for all smooth function V:

$$V(foL_g) = V(f)oL_g$$

where L_g dentotes the left multiplication by g, that is $L_g g' = g.g'$.
By an abuse of notations, we will still denote the left-invariant vector field with the same letter than the matrix from which it is built: for f a smooth function on $\mathbb{G}$ and g a point of $\mathbb{G}$ and M a matrix of $\mathfrak{g}$, we set

$$M(f)(g) = \lim_{t \to 0} \frac{1}{t} \left(f(g.e^{tX}) - f(g) \right).$$

A matrix M of $\mathfrak{g}$ also induces a right invariant vector field, which we denote by $\hat{M}$ and which is given by

$$\hat{M}(f)(g) = \lim_{t \to 0} \frac{1}{t} \left(f(e^{tX}.g) - f(g) \right)$$

for f a smooth function on $\mathbb{G}$ and g a point of $\mathbb{G}$.

2.1.1 The Heisenberg group $\mathbb{H}$

The Heisenberg group $\mathbb{H}$ is the group of 3×3 matrices:

$$\begin{pmatrix} 1 & x & \frac{z}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

This group is non commutative and the law of the group is polynomial and can be written in $\mathbb{R}^3$:

$$(x, y, z).(x', y', z') = (x + x', y + y', z + z' + 2xy').$$

The Lie algebra of $\mathbb{H}$ is spanned by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which the following relations hold

$$[X, Y] = 2Z, \ [X, Z] = [Y, Z] = 0.$$

The center of the Lie algebra is spanned by the matrix Z and the Lie algebra is nilpotent of order 2 in the sense that the iterated brackets $[A, [B, C]]$ vanish for all $A, B, C \in \mathfrak{h}$. Thus we obtain:

Proposition 2.1.1. *If A and B are in $\mathfrak{h}$,*

$$\exp(A) \exp(B) = \exp(A + B + \frac{1}{2}[A, B]) \tag{2.1.1}$$

This relation is not so senseless even if it can be very easily proved with a little computation. It is indeed coming from the Baker-Campbell-Hausdorff formula which express the product of the exponential of two matrices as the exponential of some quantity. To be more precise, for two matrices M and N:

$$\exp(M) \exp(N) = \exp(P(M, N))$$

where $P(M, N)$ is a Lie serie which only depends on the iterated brackets of M and N:

$$P(M, N) = M + N + \frac{1}{2}[M, N] + \frac{1}{12}[[M, N], N] - \frac{1}{12}[[M, N], M] + \dots \tag{2.1.2}$$

In the case of the Heisenberg group whose Lie algebra is nilpotent of order 2, this serie stops after the first bracket term.
We prefer to work with the exponential coordinates, that is the coordinates in the Lie algebra. We identify then $g \in \mathbb{H}$ with the triple $(x, y, z) \in \mathbb{R}^3$ such that $g = \exp(xX + yY + zZ)$. The group law in these coordinates becomes:

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + (xy' - yx')) \tag{2.1.3}$$

and the inverse of an element is:

$$(x, y, z)^{*-1} = (-x, -y, -z). \tag{2.1.4}$$

The expressions of the left-invariant vector fields in these exponential coordinates are then:

$$X = \partial_x - y\partial z,$$
$$Y = \partial_y + x\partial z$$

and

$$Z = \partial_z.$$

Whereas the right-invariant vector fields write:

$$\hat{X} = \partial_x + y\partial z,$$
$$\hat{Y} = \partial_y - x\partial z$$

and

$$\hat{Z} = \partial_z.$$

2.1.2 The Lie group $\mathbf{SU}(2)$

The Lie group $\mathbf{SU}(2)$ is the group of 2×2, complex, unitary matrices of determinant 1, that is the matrices of the form

$$\left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix}, z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1 \right\}.$$

It is a compact Lie group which clearly can be identified with the 3-dimensional sphere S^3. Its Lie algebra $\mathfrak{su}(2)$ consists of 2×2, complex, skew-adjoint matrices of trace 0. A basis of $\mathfrak{su}(2)$ is formed by the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

for which the following relations hold

$$[X, Y] = 2Z, \quad [X, Z] = -2Y, \quad [Y, Z] = 2X. \tag{2.1.5}$$

The flows of the vector fields from the identity write:

$$e^{tX} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \ e^{tY} = \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix}, \ e^{tZ} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

And therefore, in the coordinates (x_1, y_1, x_2, y_2) where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the left-invariant vector fields induced are given by:

$$X = \begin{pmatrix} -x_2 \\ -y_2 \\ x_1 \\ y_1 \end{pmatrix}, \ Y = \begin{pmatrix} -y_2 \\ x_2 \\ -y_1 \\ x_1 \end{pmatrix} \text{ and } Z = \begin{pmatrix} -y_1 \\ x_1 \\ y_2 \\ -x_2 \end{pmatrix}.$$

The above notation just means that $X = -x_2\partial_{x_1} - y_2\partial_{y_1} + x_1\partial_{x_2} + y_1\partial_{y_2}$. It is then immediate to see that the vector fields X, Y and Z form in each point of S^3 an orthonormal basis of the tangent space and that the canonical Laplace-Beltrami operator on S^3 writes:

$$\Delta_{S^3} = X^2 + Y^2 + Z^2.$$

2.1.3 The Lie group $\mathbf{SL}(2,\mathbb{R})$

The Lie group $\mathbf{SL}(2,\mathbb{R})$ is the group of 2×2, real matrices of determinant 1, that is the of matrices:

$$\left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right), a,b,c,d,\in\mathbb{R}, ad-bc=1\right\}.$$

Its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of 2×2 matrices of trace 0. A basis of $\mathfrak{sl}(2,\mathbb{R})$ is formed by the matrices:

$$X=\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right),\ Y=\left(\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array}\right),\ Z=\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right),$$

for which the following relationships hold

$$[X,Y]=2Z,\quad [X,Z]=2Y,\quad [Y,Z]=-2X. \tag{2.1.6}$$

The flows of the vector fields from the identity write:

$$e^{tX}=\left(\begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array}\right),\ e^{tY}=\left(\begin{array}{cc} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{array}\right),\ e^{tZ}=\left(\begin{array}{cc} \cos t & -\sin t \\ \sin t & \cos t \end{array}\right).$$

And therefore the left invariant-vector fields induced are given in this coordinates (a,b,c,d) by:

$$X=\left(\begin{array}{c} a \\ -b \\ c \\ -d \end{array}\right),\ Y=\left(\begin{array}{c} -b \\ -a \\ -d \\ -c \end{array}\right) \text{ and } Z=\left(\begin{array}{c} b \\ -a \\ d \\ -c \end{array}\right).$$

2.1.4 The 3 model spaces in a same framework

In what follows, we introduce precisely the subelliptic structure which we will study in all the sequel. We also see that, by introducing a parameter ρ, that we can deal in the same time with our three model spaces. This parameter shall have a curvature meaning.
Thus we shall consider a three-dimensional Lie group $\mathbf{G}$ with Lie algebra $\mathfrak{g}$ and we assume that there is a basis $\{X,Y,Z\}$ of $\mathfrak{g}$ such that

$$[X,Y]=2Z$$
$$[X,Z]=-2\rho Y$$
$$[Y,Z]=2\rho X$$

where $\rho\in\mathbb{R}$.
As we have seen before, we can choose the $\mathbf{SU}(2)$ group for $\rho=1$, the Heisenberg group for $\rho=0$ and the $\mathbf{SL}(2,\mathbb{R})$ group for $\rho=-1$. We consider on the Lie group $\mathbf{G}$ the second order differential operator

$$L=X^2+Y^2$$

where as before X and Y denote the left-invariant vector fields generated by the matrices X and Y. This operator is then left-invariant. This operator is clearly not elliptic. But according to the relations of the Lie algebra and Hormander's theorem, it is hypoelliptic. Associated to L, there is a notion of distance given

$$\delta(g_1, g_2) = \sup_{f \in \mathcal{C}} \{ | f(g_1) - f(g_2) | \}$$

where $\mathcal{C}$ is the set of smooth maps $\mathbb{G} \to \mathbb{R}$ that satisfy $(Xf)^2 + (Yf)^2 \leq 1$. This distance corresponds in fact to the Carnot-Carathéodory distance. Via Chow's theorem, the Carnot-Carathéodory distance can also be defined as the minimal length of horizontal curves joining two given points (see Chapter 3 of [15]). Therefore the geometry associated to L is not Riemannian but only subriemannian.

We also consider the heat semigroup $P_t = e^{tL}$ on $\mathbb{G}$. In the introduction, we justified the existence of such a semigroup. By hypoellipticity, the heat semigroup $P_t = e^{tL}$ admits a smooth kernel with respect to the Haar measure μ of $\mathbb{G}$ and it is positive everywhere. We will denote it by p_t on the three groups and sometimes by h_t on the Heisenberg group. This means therefore that, for a smooth function f on $\mathbb{G}$ and $g \in \mathbb{G}$:

$$P_t(f)(g) = \int_{\mathbb{G}} p_t(g, g') f(g') d\mu(g').$$

By the left invariance of our models, the semigroup P_t commutes with the left translations, that is, if $l_g(f)(g') = f(g.g')$:

$$P_t(f)(g) = l_g P_t(f)(0) = P_t(l_g f)(0).$$

Using μ is a left-invariant Haar measure, this implies that

$$p_t(g, g') = p_t(0, g^{-1}.g').$$

2.1.5 The Γ_2 formalism

We shall often make use of the following notations (see [4], [5]): We set for f, g smooth functions,

$$2\Gamma(f, g) = L(fg) - fLg - gLf$$

and

$$2\Gamma_2(f, g) = L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf).$$

In the Riemannian case, where L is an elliptic second order operator on a smooth manifold, with no constant term, $\Gamma(f, g)$ stands for $\nabla f \cdot \nabla g$.

In the present setting,

$$\Gamma(f, f) = (Xf)^2 + (Yf)^2$$

and

$$\Gamma_2(f, f) = (X^2 f)^2 + (Y^2 f)^2 + \frac{1}{2}\left((XY + YX)f\right)^2 + 2(Zf)^2 \tag{2.1.7}$$

$$+ 4\rho\Gamma(f, f) - 4(Xf)(YZf) + 4(Yf)(XZf). \tag{2.1.8}$$

The computation for the Γ is easy and classical. Let us see how the computation works for the Γ_2, we have:

$$\begin{aligned} \Gamma_2(f, f) &= \frac{1}{2} L\left(\Gamma(f)\right) - \Gamma(f, Lf) \\ &= \frac{1}{2} L\left((Xf)^2 + (Yf)^2)\right) - (Xf)X(Lf) - Y(f)Y(L). \end{aligned}$$

But by definition:

$$L(g^2) = 2gLg + 2\Gamma(g,g),$$

thus

$$\begin{aligned}\Gamma_2(f,f) &= (X^2f)^2 + (Y^2f) + (XYf)^2 + (YXf)^2 \\ &+ X(f)[L,X](f) + Y(f)[L,Y](f).\end{aligned}$$

Now we write:

$$(XYf)^2 + (YXf)^2 = \frac{1}{2}((XY-YX)f)^2 + \frac{1}{2}((XY+YX)f)^2 = 2(Zf)^2 + \frac{1}{2}((XY+YX)f)^2$$

and

$$\begin{aligned}[L,X] &= [Y^2,X] \\ &= Y[Y,X] + [Y,X]Y \\ &= -2YZ - 2ZY \\ &= 2[Y,Z] - 4YZ \\ &= 4\rho X - 4YZ.\end{aligned}$$

Similarly we write

$$\begin{aligned}[L,Y] &= [X^2,Y] \\ &= X[X,Y] + [X,Y]X \\ &= 2XZ + 2ZX \\ &= -2[X,Z] + 4XZ \\ &= 4\rho Y + 4XZ\end{aligned}$$

which ends our computation when we put everything together.

2.2 The different notions of curvature fail

In this section, we see that the known notions of a Ricci curvature bounded from below fail. First, the Ricci curvature for L is not defined but we can approximate the operator L by the elliptic operators $\Delta_\varepsilon = X^2 + Y^2 + \varepsilon^2 Z^2$. For these operators Δ_ε, the Ricci curvature is well defined but we will see the lower bound on the Ricci curvature goes to $-\infty$ when ε goes to 0. Indeed, the elliptic operator Δ_ε is the Laplace-Beltrami operator of the Riemanian manifold $\mathbb{G}$ whose metric g_ε is given by setting $(X, Y, \varepsilon Z)$ as an orthonormal basis. One can then compute the Levi-Civita connection using the Kozul formula:

$$\begin{aligned}2g_\varepsilon(\nabla_U V, W) &= U.g_\varepsilon(V,W) + V.g_\varepsilon(U,W) - W.g_\varepsilon(U,V) \\ &+ g_\varepsilon([U,V],W) - g_\varepsilon([U,W],V) - g_\varepsilon([V,W],U)\end{aligned}$$

to get

$$\nabla_X X = 0,\ \nabla_Y X = -\frac{1}{\varepsilon}(\varepsilon Z),\ \nabla_{\varepsilon Z} X = \left(-\frac{1}{\varepsilon} + 2\rho\varepsilon\right) Y;$$

$$\nabla_X Y = \frac{1}{\varepsilon}(\varepsilon Z),\ \nabla_Y Y = 0,\ \nabla_{\varepsilon Z} Y = \left(\frac{1}{\varepsilon} - 2\rho\varepsilon\right) X;$$

$$\nabla_X(\varepsilon Z) = -\frac{1}{\varepsilon} Y,\ \nabla_Y(\varepsilon Z) = \frac{1}{\varepsilon} X,\ \nabla_{\varepsilon Z}(\varepsilon Z) = 0.$$

Now the Riemannian curvature tensor is given by

$$R(U,V)W = -\left(\nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W\right)$$

and the Ricci tensor by the trace of the endomorphism:

$$Ric(U,V) = trace(W \to R(W,U)V).$$

Computations from the expression of the Levi-Civita connection gives us that, in the orthonormal basis $(X, Y, \varepsilon Z)$, the Ricci tensor writes:

$$\begin{pmatrix} 4\rho - \frac{2}{\varepsilon^2} & 0 & 0 \\ 0 & 4\rho - \frac{2}{\varepsilon^2} & 0 \\ 0 & 0 & \frac{2}{\varepsilon^2} \end{pmatrix} \tag{2.2.9}$$

and as we said the lower bound on the Ricci curvature goes to $-\infty$ when ε goes to 0.

Remark 2.2.1. *When $\rho = 1$ and $\varepsilon = 1$, the Ricci tensor equals $2Id$. This is consistant with the fact that the corresponding operator $X^2 + Y^2 + Z^2$ is the Laplace-Beltrami over the unit sphere of dimension 3 since the Ricci tensor of the unit sphere of dimension n is $(n-1)Id$.*

There is another concept of curvature known as the Bakry-Emery criterion. This criterion is an extension of the notion of the Ricci curvature bounded from below on a Riemannian manifold. The Bakry-Emery criterion $CD(\rho, N)$ for the operator L writes

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{N}(Lf)^2$$

for any smooth function f; and the Bakry-Emery criterion $CD(\rho, \infty)$ for the operator L writes

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f).$$

for any smooth function f. In a case where L is the Laplace-Beltrami operator on a Riemannian manifold (of dimension N) the criterions $CD(\rho, N)$ and $CD(\rho, \infty)$ are equivalent with the fact that the Ricci cuvature of the manifold is bounded from below by ρId. Here, the objects are well defined in our subelliptic setting, but we do not have any $CD(\rho, \infty)$ inequalities. One way to see it, is to obtain it on the Heisenberg group by taking a smooth function with compact support which equals t^2 around the origin (see [60]) and then to obtain it for the other spaces using the fact that their tangent space in a point is the Heisenberg group (see Proposition 4.7.2 and Corollary 4.7.4). Another way to do it is to use the equivalence with a lower bound on the Ricci curvature and the previous computations of this lower bound for the operators Δ_ε.

Another extension of the notion of the Ricci curvature bounded from below to metric spaces was done independently by Lott and Villani [73] and Sturm [85, 86]. Their notion is based on convexity properties of the geodesics in the Wasserstein space (the Wasserstein space is the set of probabilities on the manifold which admit a finite second moment). Juillet in [61] show that such a criterion also fails on the Heisenberg group. Using the convergence towards their tangent space, this criterion also should not be satisfied on $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$. More precisely, Juillet shows that a consequence of their notion: the Brunn-Minkowski inequality was not satisfied on the Heisenberg group. Note also from the work [24] that this Brunn-Minkowski inequality can also be taken as a generalisation of the notion of the Ricci curvature bounded from below.

The only known curvature criterion for our model spaces are some properties of contraction of the measure, namely some $MCP(\rho;2,3)$ properties (see [61] and [2]). Note that $MCP(0;2,3)$ is just the "classical" measure contraction property $MCP(0,5)$ (see [86] and [79] for a study of this criterion).

2.3 Submersions

In this section we will see a framework in which the subelliptic structures that we had just described arise in a natural way.

2.3.1 The Hopf fibration on $\mathbf{SU}(2)$

There is a very deep link between the subelliptic structure we present on $\mathbf{SU}(2)$ and the Hopf fibration on the three dimensional sphere S^3. In this subsection we will give two different ways to see it. The first one is based on the complex projective projection and can be generalize to higher dimensions. The second one is based on the action of the quaternions over the 2-dimensional sphere and as we will see it in the next subsection, this way can be generalized for the $\mathbf{SL}(2,\mathbb{R})$ group.

First recall that the spaces $\mathbf{SU}(2)$ and the three dimensional sphere S^3 of radius 1 are diffeomorphic. Thus we can look at the 3-dimensional sphere embedded in $\mathbb{C}^2$, the Hopf fibration Π is then just the restriction to S^3 of the complex projective projection. Therefore it writes:

$$S^1 \to S^3 \to S^2. \tag{2.3.10}$$

Note that in this writing, S^2 identifies with the complex projective space $\mathbb{C}P_1$ and for some reasons that we will explain later we consider S^2 as the 2 dimensional sphere of radius $\frac{1}{2}$.
More generally, the Hopf fibration can be defined for the sphere of dimension $2n+1$ by looking at it embedded in $\mathbb{C}^{n+1}$ and taking the restriction of the complex projective projection, it writes:

$$S^1 \to S^{2n+1} \to \mathbb{C}P_n. \tag{2.3.11}$$

It is easy to see that all the points which have the same image are in a same great circle of S^{2n+1}. Indeed they belong to the intersection of the sphere with a complex plane. In other words it means the fibers of this projection Π are great circles and more precisely they are

$$\{(e^{it}z_1, e^{it}z_2, \dots, e^{it}z_{n+1}), t \in [0,2\pi]\}.$$

Let us denote by V the vector field on S^{2n+1} generated by the action:

$$Vf(z_1,\dots,z_n) = \lim_{t\to 0} \frac{1}{t}\left(f(e^{it}z_1, e^{it}z_2, \dots, e^{it}z_{n+1}) - f(z_1,\dots,z_n)\right)$$

for f a smooth function on S^{2n+1}. This vector V is called a vertical vector field. The set of horizontal vector fields $\mathcal{H}$ is then the set of vector fields orthogonal to V with repect to the classical Riemannian metric on the sphere S^{2n+1}. This Hopf fibration is also a Riemannian submersion, that means that the restiction of the differential of Π to the horizontal vector fields is an isometry, i.e.

$$d\Pi|_{\mathcal{H}_x} : H_x \to T_\Pi(x)\mathbb{C}P_n$$

is an isometry. In fact, to obtain really an isometry, we have to choose a good dilation of the space $\mathbb{C}P_n$.

Now what we are going to do is to see is how the above statements express precisely in our setting. We treat here only the case of the dimension 3, but this should generalize to all odd dimension spheres. What we will see is that, in our setting, the vector field Z coincides exactly with the vertical vector field V, that the vector fields X and Y form an orthonormal basis in each point of the horizontal tangent vectors and that the classical Laplace-Beltrami on the 3-sphere writes

$$\Delta_{S^3} = X^2 + Y^2 + Z^2 = L + Z^2$$

whereas the classical Laplace-Beltrami on the 2-sphere of radius 1 can be written:

$$\Delta_{S^2} = \frac{1}{4}\left((d\Pi X)^2 + (d\Pi Y)^2\right).$$

The constant $\frac{1}{4}$ is coming since we deal in the Hopf fibration with the sphere of radius $\frac{1}{2}$.

In order to use the classical expressions of the Hopf fibration, we choose the following particular identification between the $\mathbf{SU}(2)$ group and the 3-sphere S^3: the point

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix}$$

is identified with the point $(x_1, y_1, x_2, -y_2)$ of the 3-sphere, where $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$. In these new coordinates the vector fields X, Y and Z just write:

$$X = \begin{pmatrix} -x_2 \\ y_2 \\ x_1 \\ -y_1 \end{pmatrix}, Y = \begin{pmatrix} y_2 \\ x_2 \\ -y_1 \\ -x_1 \end{pmatrix} \text{ and } Z = \begin{pmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \end{pmatrix}.$$

The Hopf projection Π is given by the map

$$(z_1, z_2) \in \mathbb{C}^2 \to \left(\frac{1}{2}(|z_1|^2 - |z_2|^2), z_1\bar{z_2}\right) \in \mathbb{R} \times \mathbb{C}$$

or equivalently by the map

$$(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \to \left(\frac{1}{2}(x_1^2 + y_1^2 - x_2^2 - y_2^2), x_1x_2 + y_1y_2, y_1x_2 - y_2x_1\right) \in \mathbb{R}^3.$$

To see that the image of the Hopf fibration restricted to S^3 is the sphere S^2 of radius $\frac{1}{2}$, note that, if (z_1, z_2) belongs to S^3

$$\begin{aligned}(|z_1|^2 - |z_2|^2)^2 + |2z_1\bar{z_2}|^2 &= |z_1|^4 + |z_2|^4 - 2|z_1||z_2| + 4|z_1||z_2| \\ &= (|z_1|^2 + |z_2|^2)^2 \\ &= 1.\end{aligned}$$

Now let us compute the differential of this application, we have:

$$d\Pi = \begin{pmatrix} x_1 & y_1 & -x_2 & -y_2 \\ x_2 & y_2 & x_1 & y_1 \\ -y_2 & x_2 & y_1 & -x_1 \end{pmatrix}$$

Therefore the images of the vector fields X, Y and Z are:

$$d\Pi.X = \begin{pmatrix} -2x_1x_2 + 2y_1y_2 \\ x_1^2 - y_1^2 - x_2^2 + y_2^2 \\ 2x_1y_1 + 2x_2y_2 \end{pmatrix},$$

$$d\Pi.Y = \begin{pmatrix} 2x_1y_2 + 2y_1x_2 \\ -2x_1y_1 + 2x_2y_2 \\ x_1^2 - y_1^2 + x_2^2 - y_2^2 \end{pmatrix}$$

and

$$d\Pi.Z = 0.$$

Now one can check that $d\Pi.X$ and $d\Pi.Y$ form an orthonormal basis in each point of the tangent space of the sphere S^2. Note that the metric one S^2 is the one inherited from the restriction of the metric of $\mathbb{R}^3$. Let us check $d\Pi.X$ is of norm 1:

$$\begin{aligned} & (-2x_1x_2 + 2y_1y_2)^2 + (2x_1y_1 + 2x_2y_2)^2 + (x_1^2 - y_1^2 - x_2^2 + y_2^2)^2 \\ = \; & 4x_1^2x_2^2 + 4y_1^2y_2^2 + 4x_1^2y_1^2 + 4x_2^2y_2^2 \\ & + x_1^4 + y_1^4 + x_2^4 + y_2^4 \\ & - 2x_1^2x_2^2 - 2x_1^2y_1^2 + 2x_1^2y_2^2 \\ & + 2y_1^2x_2^2 - 2y_1^2y_2^2 \\ & - 2x_2^2y_2^2 \\ = \; & x_1^4 + y_1^4 + x_2^4 + y_2^4 \\ & + 2x_1^2x_2^2 + 2x_1^2y_1^2 + 2x_1^2y_2^2 \\ & + 2y_1^2x_2^2 + 2y_1^2y_2^2 \\ & + 2x_2^2y_2^2 \\ = \; & (x_1^2 + y_1^2 + x_2^2 + y_2^2)^2 \\ = \; & 1 \end{aligned}$$

if the point (x_1, y_1, x_2, y_2) belongs to S^3. The two other computations $||d\Pi.Y||_2 = 1$ and $\langle d\Pi.X, d\Pi.Y\rangle = 0$ are very similar.

The above representation is interesting because, as we saw it, it makes sense for all odd dimension bigger than 3 and not only for dimension 3. This implies that the methods of chapter 4 can be applied to obtain the spectral decomposition and the integral representation of the subelliptic heat kernel in these higher dimensional settings.

However, there is another representation of the Hopf fibration which is obtained by using the quaternions. Let us express it. The idenfication between the $\mathbf{SU}(2)$ group and the unit quaternions can be done in the following way: to a point

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix}$$

of the $\mathbf{SU}(2)$ group is associated the unit quaternion $q = w + ix + jy + kz$, where

$$w = x_1, x = x_2, y = y_2, z = y_1$$

and with $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$.
It is easy to see that this identification preserves the group law and therefore is a group isomorphism between $\mathbf{SU}(2)$ and the groups of unit quaternions.
In these new coordinates (w, x, y, z) the vector fields X, Y and Z write:

$$X = \begin{pmatrix} -x \\ -y \\ w \\ z \end{pmatrix}, Y = \begin{pmatrix} -y \\ x \\ -z \\ w \end{pmatrix} \text{ and } Z = \begin{pmatrix} -z \\ w \\ y \\ x \end{pmatrix}.$$

We also interpret a point $(y_1, y_2, y_3) \in \mathbb{R}^3$ as the imaginary quaternion

$$p = iy_1 + jy_2 + ky_3.$$

Then, for each unit quaternion q we can consider the linear mapping:

$$r_q : p \in R^3 \to qpq^* \in \mathbb{R}^3$$

It is well known since Cayley that, for each unit quaternion q, this mapping r_q is a rotation of $\mathbb{R}^3$. It is indeed easy to see it is an isometry since:

$$|qpq^*|^2 = qpq^*qpq^* = qpp^*q^* = |p|^2qq* = |p|^2.$$

One can also compute explicitly the rotation induced by a unit quaternion $q = w + ix + jy + kz$; it is given by the orthogonal matrix:

$$\begin{pmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{pmatrix}$$

In fact, this mapping $q \to r_q$ identifies the group of the unit quaternions with the group of rotations of $\mathbb{R}^3$ modulo the fact that two opposite unit quaternions q and $-q$ determines the same rotation. As the rotations of $\mathbb{R}^3$ acts transitively on the sphere S^2, we obtain a transitive action of the unit quaternions over the 2-sphere S^2. One concrete way to define the Hopf fibration in this setting is to fix an imaginary quaternion p, for example $p = k$ which correponds to the point $(0, 0, 1 =$ on the 2-sphere, and to look at its image by the rotations r_q. We obtain then the map $P : q \to qkq^*$ which sent a unit quaternion to a point of S^2. In coordinates, this map can be written

$$P : (w, x, y, z) \to (2(xz + wy), 2(yz - wx), 1 - 2(x^2 + y^2))$$

It is a smooth map and its differential writes:

$$dP = \begin{pmatrix} 2y & 2x & 2z & 2w \\ -2w & 2y & -2w & 2z \\ 0 & 0 & -4x & -4y \end{pmatrix}.$$

And one has:

$$dP.X = \begin{pmatrix} -4xy + 4wz \\ 2x^2 - 2y^2 - 2w^2 + 2z^2 \\ -4xw - 4yz \end{pmatrix},$$

$$dP.Y = \begin{pmatrix} 2x^2 - 2y^2 + 2w^2 - 2z^2 \\ +4xy + 4wz \\ +4xz - 4yw \end{pmatrix}$$

and

$$dP.Z = 0.$$

One can then check that as before $(\frac{1}{2}dP.X, \frac{1}{2}dP.Y)$ form an orthonormal basis in each point of the tangent space of the sphere S^2 of radius 1.
We can also describe the fibers of this bundle. The fiber for a point (a, b, c) of S^2 consist of all the unit quaternions whose associated rotation sends the point $(0, 0, 1)$ on this point (a, b, c). These fibers are great circle of S^3. For example the unit quaternions that fix in this action the point $(0, 0, 1)$ are the ones of the form $q_\theta = \cos\theta + k \sin\theta$ for $\theta \in [0, 2\pi]$. It is interesting to note that the rotation induced by q_θ is the rotation by -2θ around the z-axis. Indeed it writes:

$$\begin{pmatrix} 1 - 2\sin^2\theta & -2\cos\theta\sin\theta & 0 \\ 2\cos\theta\sin\theta & 1 - 2\sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The other fibers can be obtained by using the multpilication of quaternions which transforms into the compositions of rotations. If (a, b, c) is a point of the 2-sphere different form the antipodal point $(0, 0, -1)$, then $Pq_{(a,b,c)} = (a, b, c)$ if $q_{(a,b,c)}$ is the unit quaternion

$$q_{(a,b,c)} = \frac{1}{\sqrt{2(1+c)}}(1 + c - ib + ja).$$

Therefore for such a point (a, b, c) of S^2 the fiber is given by the quaternions of the form $q_{(a,b,c)} \times q_\theta$, $\theta \in [0, 2\pi]$, that is the points of S^3 of the form

$$\frac{1}{\sqrt{2(1+c)}} \left((1 + c)\cos\theta, a\sin\theta - b\cos\theta, a\cos\theta + b\sin\theta, (1 + c)\sin\theta\right).$$

The last fiber for the point $(0, 0, -1)$ can be obtained by noticing that $Pq_{(0,0,-1)} = (0, 0, -1)$ if $q_{(0,0,-1)} = i$ and it produces the fiber

$$(0, \cos\theta, -\sin\theta, 0)$$

which completes the bundle.

2.3.2 A generalization of the Hopf fibration for $\mathbf{SL}(2,\mathbb{R})$

The generalisation of the Hopf fibration for the $\mathbf{SL}(2,\mathbb{R})$ group follows the same line as the representation of the Hopf fibration for $\mathbf{SU}(2)$ using the quaternions. This time we let act the $\mathbf{SL}(2,\mathbb{R})$ on the 2-dimensional hyperbolic space and it writes:

$$S^1 \to \mathbf{SL}(2,\mathbb{R}) \to H^2.$$

Here, we look at H^2 as the Poincaré hyperbolic upper-plane and we consider the natural action of $\mathbf{SL}(2,\mathbb{R})$ on it. As in the case of $\mathbf{SU}(2)$, this is an action by isometries of H^2 and it is transitive. The action is given by the homographies:

$$R_M : z \in H^2 \to \frac{az+b}{cz+d} \text{ for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ a point of } \mathbf{SL}(2,\mathbb{R}).$$

Note that the matrices M and $-M$ induced the same homography. To obtain the fibration explicitly, we look a the image of a particular point of H^2 for example the point i. We obtain then the map:

$$\phi M_{a,b,c,d} \in \mathbf{SL}(2,\mathbb{R}) \to \frac{ai+b}{ci+d} = \frac{bd+ac}{c^2+d^2} + i\frac{1}{c^2+d^2} \in H^2.$$

We can also consider the image to be $\mathbb{R} \times \mathbb{R}_{>0}$. The map ϕ is smooth and its differential is then

$$d\phi = \frac{1}{c^2+d^2} \begin{pmatrix} c & d & a - \frac{2c(bd+ac)}{c^2+d^2} & b - \frac{2d(bd+ac)}{c^2+d^2} \\ 0 & 0 & -\frac{2c}{c^2+d^2} & -\frac{2d}{c^2+d^2} \end{pmatrix}.$$

Recall that, in these coordinates on $\mathbf{SL}(2,\mathbb{R})$, the vector fields X, Y and Z write:

$$X = \begin{pmatrix} a \\ -b \\ c \\ -d \end{pmatrix}, Y = \begin{pmatrix} -b \\ -a \\ -d \\ -c \end{pmatrix} \text{ and } Z = \begin{pmatrix} b \\ -a \\ d \\ -c \end{pmatrix}.$$

After some computations, one obtains:

$$d\phi.X = \frac{2}{(c^2+d^2)^2} \begin{pmatrix} 2cd \\ d^2-c^2 \end{pmatrix},$$

$$d\phi.Y = \frac{2}{(c^2+d^2)^2} \begin{pmatrix} c^2-d^2 \\ 2cd \end{pmatrix},$$

and

$$d\phi.Z = 0.$$

One can then check that as before $(\frac{1}{2}dP.X, \frac{1}{2}dP.Y)$ form an orthonormal basis in each point of the tangent space of the Poincaré half-upper plane H^2. Note that the metric on H^2 is given by $g_{(x,y)} = \frac{dx^2+dy^2}{y^2}$. The fiber of the point i is given by the matrices $M_{a,b,c,d}$ such that $c^2+d^2=1$, $bd+ac=0$ and of course $ad-bc=1$, that is the rotation M_θ where

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Note that, as $M_{\theta+\pi} = -M_\theta$, the rotations R_{M_θ} and $R_{M_{\theta+\pi}}$ are the same. This fiber form the maximal compact subgroup K of $\mathbf{SL}(2,\mathbb{R})$.
This fibration can be seen as a Riemannian submersion from $\mathbf{SL}(2,\mathbb{R})$ equipped with the metric $g_R = dX^2+dY^2+dZ^2$ over the hyperbolic space of dimension 2 with the canonical metric but the Riemannian metric $g_R = dX^2+dY^2+dZ^2$ does not seem to have anything canonical. Instead it seems more interseting to look at it as a pseudo-Riemannian submersion. We equip the $\mathbf{SL}(2,\mathbb{R})$ group with the pseudo-Riemanian metric $g_{pR} = dX^2+dY^2-dZ^2$. By the above computations it is clear that the fibration is then a pseudo-Riemannian submersion from $\mathbf{SL}(2,\mathbb{R})$ with this pseudo-Riemanian metric over the hyperbolic space of dimension 2 with the canonical metric. The vertical tangent space associated is spanned by the the vector field Z and the vector fields X and Y form an orthonormal basis in each point of the horizontal tangent vectors. Note that the pseudo-Riemannian metric restricted to the horizontal vector fields is positive definite and coincides with our subelliptic metric.
What is interesting in this setting is that this time the metric $g_{pR} = dX^2+dY^2-dZ^2$ is canonical since it is the one inherited from the Killing form. The associated operator is the Casimir operator $\Box = X^2+Y^2-Z^2$. As the canonical Laplace-Beltrami operator on $\mathbf{SU}(2)$, the Casimir operator belongs to the center of the envelopping algebra and in fact generates it (see [87]). Therefore, in our setting, we have the following relations between the Casimir operator $\Box$ and our sublaplacian L on $\mathbf{SL}(2,\mathbb{R})$:

$$\Box = X^2+Y^2-Z^2 = L - Z^2$$

Moreover the classical Laplace-Beltrami on the 2-dimensional hyperbolic space H_2 can be written:

$$\Delta_{H_2} = \frac{1}{4}\left((d\phi X)^2 + (d\phi Y)^2\right).$$

Remark 2.3.1. *In both cases fo $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, the above submersions are double covering maps. The $\mathbf{SU}(2)$ group is simply connsected but this is not true for the $\mathbf{SL}(2,\mathbb{R})$ group since it is homeomorphic to $\mathbb{R}^2 \times S^1$. Maybe it would have been more interesting to study the universal covering $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ of $\mathbf{SL}(2,\mathbb{R})$ to obtain really a model space.*

2.4 The dilation structure on $\mathbb{H}$

In this section we will see that the Heisenberg group admits a non-isotropic dilation. This dilation structure will be useful in all the sequel and it gives a first justification why we call the Heisenberg group a flat space.
The dilation structure is given by the dilation vector field D which reads in the exponential coordinates:

$$D = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + z\partial_z.$$

This vector fields satisfies the fundamental relation

$$[L,D] = L. \tag{2.4.12}$$

Indeed, in these coordinates, L just writes

$$L = \partial^2_{xx} + \partial^2_{yy} + 2(x\partial_y - y\partial_x)\partial_z + (x^2+y^2)\partial^2_{zz}.$$

Let $(T_t)_{t\geq 0}$ be the group of dilations generated by D, it is given by

$$T_t(f)(x,y,z) = f(e^{t/2}x, e^{t/2}y, e^t z).$$

Let us also denote by dil_λ the dilation in the group given by:

$$dil_\lambda(x,y,z) = (\lambda x, \lambda y, \lambda^2 z)$$

so that

$$T_t(f)(x,y,z) = f(dil_{e^{t/2}}(x,y,z)).$$

Using (2.4.12), one obtain that for a smooth compactly supported function f,

$$\begin{aligned} \frac{d}{dt} T_{-t} L T_t(f)(x) &= T_{-t}[L,D]T_t(f)(x) \\ &= T_{-t} L T_t(f)(x) \end{aligned}$$

and therefore as in $t=0$, $T_{-t}LT_t(f)(x)$,

$$T_{-t}LT_t(f)(x) = e^t Lf(x).$$

Thus, one has

$$T_{-t}LT_t = e^t L$$

or equivalently

$$LT_t = e^t T_t L. \tag{2.4.13}$$

As a consequence,

$$\begin{aligned} \Gamma(T_t f, T_t g) &= \frac{1}{2} L(T_t(fg)) - (T_t f)(LT_t g) - (T_t g)(LT_t f) \\ &= e^t \left(\frac{1}{2} T_t(L(fg)) - (T_t f)(T_t Lg) - (T_t g)(T_t Lf) \right) \\ &= e^t \Gamma(f,g) \end{aligned}$$

and therefore for $g, g' \in \mathbb{H}$

$$\begin{aligned} \delta(dil_\lambda g, dil_\lambda g') &= \sup_{\Gamma(f)\leq 1} f(dil_\lambda g, dil_\lambda g') \\ &= \sup_{\Gamma(f)\leq 1} T_{2\ln\lambda} f(g) - T_{2\ln\lambda} f(g') \\ &= \sup_{\Gamma(h)\leq \lambda^2} h(g) - h(g') \\ &= \lambda \sup_{\Gamma(h)\leq 1} h(g) - h(g') \\ &= \lambda \delta(g,g'); \end{aligned}$$

which shows our structure is a real dilation structure. Moreover, we obtain also the commutation relations

$$P_t T_s = T_s P_{e^s t} \tag{2.4.14}$$

and

$$P_tD = DP_t + tLP_t. \tag{2.4.15}$$

Indeed, for the first one, for a smooth compactly supported function f, set $\phi(t) = (P_tD - DP_t - tLP_t)(f)(x)$,

$$\phi'(t) = (P_tLD - DLP_t - LP_t - tL^2P_t)(f)(x).$$

But, using (2.4.12),

$$\begin{aligned} P_tLD - DLP_t &= P_tLD - (LD - L)P_t \\ &= L(P_tD - DP_t) + LP_t \end{aligned}$$

Thus,

$$\phi'(t) = L\phi(t).$$

Noticing $\phi(0) = 0$ and $\phi(t)$ is a function in the Schwarz space, by unicity of the heat equation in this space, $\phi(t) = 0$, and so

$$P_tD - DP_t = tP_tL.$$

For the second one, set $\phi(t) = T_{-s}P_tT_s(f)(x)$. By (2.4.15) and (2.4.13),

$$\begin{aligned} \phi'(t) &= T_{-s}[P_t, D]T_s(f)(x) \\ &= tT_{-s}LP_tT_s(f)(x) \\ &= e^s tL\phi(t). \end{aligned}$$

Then, again, by unicity of the heat equation in the Schwarz space, as $\phi(0) = f(x)$,

$$\phi(t) = P_{e^s t} f(x)$$

which implies the desired equality.

As a consequence of (2.4.14) and using that 0 is a fix point for the dilation that is $T_s g(0) = g(0)$,

$$P_t(f)(0) = P_1(T_{\ln t} f)(0).$$

Therefore, using also the left invariance, one can obtain, on the Heisenberg group, the all semigroup $(P_tf)_{t\geq 0}$ from $P_1(f)(0)$.

Another useful fact concerns the adjoint of the operator D. It is easy to see that this adjoint satisfies:

$$D^* = -D - 2 \tag{2.4.16}$$

where $2 = \frac{Q}{2}$ with Q the homogenous dimension of the Heisenberg group. This fact generalises to Carnot groups which are the nilpotent Lie groups which admit a dilation (see [15]).

2.5 The cylindrical coordinates

To study the subelliptic operator L on the Lie group $\mathbb{G}$, we will use some coordinates well-adapted to the problem. We use the following cylindrical coordinates:

$$(r, \theta, z) \to \exp\left(r \cos \theta X + r \sin \theta Y\right) \exp(zZ).$$

First notice these coordinates are made to have a very simple expression for the vector field Z in all the cases, since one has

$$Z = \frac{\partial}{\partial z}$$

Indeed,

$$(\exp(r \cos \theta X + r \sin \theta Y) \exp(zZ)) \exp(\varepsilon Z) = \exp(r \cos \theta X + r \sin \theta Y) \exp((z + \varepsilon)Z).$$

And therefore, on $\mathbb{G}$ in these coordinates, one has the following equality:

$$(r, \theta, z).(0, 0, z') = (r, \theta, z + z').$$

Note also the identity of $\mathbb{G}$ is the point of coordinate $(0, 0, 0)$ and sometimes it will be denote by 0 in the sequel. These coordinates were introduced in [35] in the case of the $\mathbf{SU}(2)$ group.

The case of the Heisenberg group

Since the vector Z belongs to the center of the Lie algebra, these coordinates are only the classical cylindrical coordinates in $\mathbb{R}^3$ of the exponential coordinates. Indeed, one has:

$$(r, \theta, z) \to \exp\left(r \cos \theta X + r \sin \theta Y\right) \exp(zZ) = (r \cos \theta, r \sin \theta, z)$$

for $r > 0, \theta \in [0, 2\pi], z \in \mathbb{R}$. The left-invariant vector fields X, Y and Z write then:

$$X = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta - r \sin \theta \partial_z$$
$$Y = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta + r \cos \theta \partial_z$$
$$Z = \partial_z.$$

Whereas the corresponding right-invariant vector fields read:

$$\hat{X} = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta + r \sin \theta \partial_z$$
$$\hat{Y} = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta - r \cos \theta \partial_z$$
$$\hat{Z} = Z = \partial_z.$$

Thus, we obtain:

$$\begin{aligned} L &= X^2 + Y^2 \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + 2 \frac{\partial^2}{\partial z \partial \theta} \end{aligned}$$

The Lebesgue measure $d\mu = r dr d\theta dz$ is an invariant and in fact also symmetric measure for L. Note that L commutes with $\frac{\partial}{\partial \theta}$,and $\frac{\partial}{\partial z}$.

Remark 2.5.1 (Probabilistic interpretation). *The computation of the left-regular representation shows that if $(X_t)_{t\geq 0}$ is the Markov process that is the matrix-valued solution of the stochastic differential equation (written in Stratonovitch form)*

$$dX_t = X_t.(X \circ dB_t^1 + Y \circ dB_t^2), \quad X_0 = 0,$$

where $(B_t^1, B_t^2)_{t\geq 0}$ is a two-dimensional Brownian motion, then, in law,

$$X_t = \exp\left(\rho_t(X\cos\theta_t + Y\sin\theta_t)\right).\exp(z_t Z), \quad t \geq 0,$$

where $(\rho_t, \theta_t, z_t)_{t\geq 0}$ solve the following stochastic differential equations (written in Itô's form):

$$d\rho_t = \frac{1}{\rho_t}dt + dB_t^1,$$

$$d\theta_t = \frac{1}{\rho_t}dB_t^2,$$

$$dz_t = \rho_t dB_t^2.$$

Here we can recover the fact that the z coordinate of this process corresponds to 2 times the area of a Brownian motion on $\mathbb{R}^2$. Indeed, the variation of the algebraic area $\mathcal{A}_t$ swept out by a curve $(\rho(t), \theta(t))$ in polar coordinates in $\mathbb{R}^2$ is given by

$$d\mathcal{A}_t = \frac{\rho(t)^2 d\theta(t)}{2}.$$

Here, since the operator

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

corresponds to the Laplacian on $\mathbb{R}^2$ in polar coordiantes, the curve $(\rho(t), \theta(t))$ is a Brownian path in $\mathbb{R}^2$. Therefore:

$$d\mathcal{A}_t = \frac{\rho(t)^2 d\theta(t)}{2} = \frac{\rho(t)dB_t^2}{2} = \frac{dz_t}{2}.$$

The case of the $\mathbf{SU}(2)$ group

The computations for the cases of $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ are more involved. Here we describe the case of the $\mathbf{SU}(2)$. We will explain some of the computations that we have done.
First the cylindrical coordinates write:

$$(r, \theta, z) \to \exp\left(r\cos\theta X + r\sin\theta Y\right)\exp(zZ) = \begin{pmatrix} \cos(r)e^{iz} & \sin(r)e^{i(\theta-z)} \\ -\sin(r)e^{-i(\theta-z)} & \cos(r)e^{-iz} \end{pmatrix},$$

with

$$0 \leq r < \frac{\pi}{2}, \ \theta \in [0, 2\pi], \ z \in [-\pi, \pi].$$

Indeed, one has

$$\begin{aligned} \exp(r\cos(\theta)X + r\sin(\theta)Y) &= \exp\begin{pmatrix} 0 & re^{i\theta} \\ -re^{-i\theta} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(r) & \sin(r)e^{i\theta} \\ -\sin(r)e^{-i\theta} & \cos(r) \end{pmatrix} \end{aligned}$$

and

$$\exp(zZ) = \begin{pmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{pmatrix}.$$

Then by taking the product, one obtains

$$\exp\left(r\cos\theta X + r\sin\theta Y\right)\exp(zZ) = \begin{pmatrix} \cos(r)e^{iz} & \sin(r)e^{i(\theta-z)} \\ -\sin(r)e^{-i(\theta-z)} & \cos(r)e^{-iz} \end{pmatrix}$$

and we can choose

$$0 \le r < \frac{\pi}{2},\ \theta \in [0,2\pi],\ z \in [-\pi,\pi].$$

Now simple but tedious computations show that in these coordinates, the left-regular representation sends the matrices X, Y and Z to the left-invariant vector fields:

$$X = \cos(-\theta+2z)\frac{\partial}{\partial r} + \sin(-\theta+2z)\left(\tan r\frac{\partial}{\partial z} + \left(\tan r + \frac{1}{\tan r}\right)\frac{\partial}{\partial\theta}\right),$$

$$Y = -\sin(2z-\theta)\frac{\partial}{\partial r} + \cos(2z-\theta)\left(\tan r\frac{\partial}{\partial z} + \left(\tan r + \frac{1}{\tan r}\right)\frac{\partial}{\partial\theta}\right),$$

$$Z = \frac{\partial}{\partial z}.$$

The right regular representation sends the matrices X, Y and Z to the right-invariant vector fields

$$\hat{X} = \cos\theta\frac{\partial}{\partial r} + \sin\theta\left(\tan r\frac{\partial}{\partial z} + \left(\tan r - \frac{1}{\tan r}\right)\frac{\partial}{\partial\theta}\right)$$

$$\hat{Y} = \sin\theta\frac{\partial}{\partial r} - \cos\theta\left(\tan r\frac{\partial}{\partial z} + \left(\tan r - \frac{1}{\tan r}\right)\frac{\partial}{\partial\theta}\right).$$

$$\hat{Z} = \frac{\partial}{\partial z} + 2\frac{\partial}{\partial\theta}.$$

We therefore obtain

$$\begin{aligned} L &= X^2 + Y^2 \\ &= \frac{\partial^2}{\partial r^2} + 2\left(\frac{1}{\tan r} - \tan r\right)\frac{\partial}{\partial r} + \tan^2 r\frac{\partial^2}{\partial z^2} \\ &+ \left(2 + \frac{1}{\tan^2 r} + \tan^2 r\right)\frac{\partial^2}{\partial\theta^2} + 2(1+\tan^2 r)\frac{\partial^2}{\partial z\partial\theta} \\ &= \frac{\partial^2}{\partial r^2} + 2\,\text{cotan}\,2r\frac{\partial}{\partial r} + \tan^2 r\frac{\partial^2}{\partial z^2} + \left(\frac{2}{\sin 2r}\right)^2\frac{\partial^2}{\partial\theta^2} + 2\tan r\left(\frac{2}{\sin 2r}\right)\frac{\partial^2}{\partial z\partial\theta} \end{aligned}$$

and

$$\begin{aligned} \Delta &= X^2 + Y^2 + Z^2 \\ &= \frac{\partial^2}{\partial z^2} + L. \end{aligned}$$

The invariant and, in fact, also symmetric measure for L is then given (up to a constant) by

$$d\mu = \frac{\sin 2r}{2}drd\theta dz.$$

Note that it coincides with the bi-invariant Haar measure of the group. We prefer not to normalize the measure, hence $\mu(\mathbf{SU}(2)) = 2\pi^2$. The choice of the constant is made to obtain a good convergence towards the Lebesgue measure of $\mathbb{R}^3$ which is the invariant measure for the Heisenberg group (see section 4.7). Note that L commutes with $\frac{\partial}{\partial\theta}$ and with $\frac{\partial}{\partial z}$.
Now, let us explain how we do the computations of the expressions of the left invariant vector fields induced by the Pauli matrices X, Y and Z. What we do is to compute the derivative in $\varepsilon = 0$ of

$$(\exp(r\cos\theta X + r\sin\theta Y)\exp(zZ))\exp(\varepsilon A)$$

for $A = X, Y, Z$. As we note it before the case of the vector field Z is very easy and we have:

$$Z = \frac{\partial}{\partial z}.$$

To obtain the right invariant vector fields, we do the same with

$$\exp(\varepsilon A)\,(\exp(r\cos\theta X + r\sin\theta Y)\exp(zZ)).$$

Let us give the details for the computation of the left invariant vector field X. The other calculations for Y are similar to the X's one.
Let us calculate X. Recall:

$$\exp(\varepsilon X) = \begin{pmatrix} \cos(\varepsilon) & \sin(\varepsilon) \\ -\sin(\varepsilon) & \cos(\varepsilon) \end{pmatrix}$$

so

$$\begin{aligned}&(\exp(r\cos\theta X + r\sin\theta Y)\exp(zZ))\exp(\varepsilon X)\\ &= \begin{pmatrix} \cos(r)\cos(\varepsilon)e^{iz} - \sin(r)\sin(\varepsilon)e^{i(\theta-z)} & \cos(r)\sin(\varepsilon)e^{iz} + \sin(r)\cos(\varepsilon)e^{i(\theta-z)} \\ -\sin(r)\cos(\varepsilon)e^{-i(\theta-z)} - \cos(r)\sin(\varepsilon)e^{-iz} & -\sin(r)\sin(\varepsilon)e^{-i(\theta-z)} + \cos(r)\sin(\varepsilon)e^{-iz} \end{pmatrix}\end{aligned}$$

This must be equal to

$$\begin{pmatrix} \cos(\tilde r)e^{i\tilde z} & \sin(\tilde r)e^{i(\tilde\theta-\tilde z)} \\ \sin(\tilde r)e^{-i(\tilde\theta-\tilde z)} & \cos(\tilde r)e^{-i\tilde z} \end{pmatrix}$$

for some $\tilde r, \tilde\theta, \tilde z$ depending on ε and which equal r, θ, z for $\varepsilon = 0$. By taking real and imaginary part of each term, we obtain a system of 4 equalities and we are only interesting in calculating the derivatives of $\tilde r, \tilde\theta, \tilde z$ in $\varepsilon = 0$. The system is:

$$\begin{cases} \cos(\tilde r)\cos(\tilde z) &= \cos(r)\cos(\varepsilon)\cos(z) - \sin(r)\sin(\varepsilon)\cos(\theta - z) \\ \cos(\tilde r)\sin(\tilde z) &= \cos(r)\cos(\varepsilon)\sin(z) - \sin(r)\sin(\varepsilon)\sin(\theta - z) \\ \sin(\tilde r)\cos(\tilde\theta - \tilde z) &= \cos(r)\sin(\varepsilon)\cos(z) + \sin(r)\cos(\varepsilon)\cos(\theta - z) \\ \sin(\tilde r)\sin(\tilde\theta - \tilde z) &= \cos(r)\sin(\varepsilon)\sin(z) + \sin(r)\cos(\varepsilon)\sin(\theta - z) \end{cases}$$

By combining these equalities, we obtain:

$$\begin{aligned}\cos^2(\tilde r) &= (\cos(r)\cos(\varepsilon)\cos(z) - \sin(r)\sin(\varepsilon)\cos(\theta - z))^2 \\ &\quad + (\cos(r)\cos(\varepsilon)\sin(z) - \sin(r)\sin(\varepsilon)\sin(\theta - z))^2 \\ &= \cos^2(r)\sin^2(\varepsilon) + \sin^2(r)\cos^2(\varepsilon) - 2\cos(r)\sin(r)\cos(\varepsilon)\sin(\varepsilon)\cos(\theta - 2z)\end{aligned}$$

By derivating and taking $\varepsilon = 0$, we get:

$$-2\cos(r)\sin(r)\partial_\varepsilon|_{\varepsilon=0}\tilde r = -2\cos(r)\sin(r)cos(\theta - 2z)$$

therefore

$$\partial_\varepsilon|_{\varepsilon=0}\tilde{r} = \cos(\theta - 2z).$$

By the same method as before:

$$\tan(\tilde{z}) = \frac{\cos(r)\cos(\varepsilon)\sin(z) - \sin(r)\sin(\varepsilon)\sin(\theta - z)}{\cos(r)\cos(\varepsilon)\cos(z) - \sin(r)\sin(\varepsilon)\cos(\theta - z)},$$

so

$$\partial_\varepsilon|_{\varepsilon=0}\tilde{z} = -\tan(r)\sin(\theta - 2z)$$

and

$$\tan(\tilde{\theta} - \tilde{z}) = \frac{\cos(r)\sin(\varepsilon)\sin(z) + \sin(r)\cos(\varepsilon)\sin(\theta - z)}{\cos(r)\sin(\varepsilon)\cos(z) + \sin(r)\cos(\varepsilon)\cos(\theta - z)}$$

so

$$\partial_\varepsilon|_{\varepsilon=0}(\tilde{\theta} - \tilde{z}) = -\frac{1}{\tan(r)}\sin(\theta - 2z)$$

Finally we get:

$$X = \cos(-\theta + 2z)\frac{\partial}{\partial r} + \sin(-\theta + 2z)\left(\tan r\frac{\partial}{\partial z} + \left(\tan r + \frac{1}{\tan r}\right)\frac{\partial}{\partial \theta}\right)$$

Remark 2.5.2 (Probabilistic interpretation)**.** *The computation of the left-regular representation shows that if $(X_t)_{t\geq 0}$ is the Markov process that is the matrix-valued solution of the stochastic differential equation (written in Stratonovitch form)*

$$dX_t = X_t.(X \circ dB_t^1 + Y \circ dB_t^2), \quad X_0 = 0,$$

where $(B_t^1, B_t^2)_{t\geq 0}$ is a two-dimensional Brownian motion, then, in law,

$$X_t = \exp\left(\rho_t(X\cos\theta_t + Y\sin\theta_t)\right).\exp(z_t Z), \quad t \geq 0,$$

where $(\rho_t, \theta_t, z_t)_{t\geq 0}$ solve the following stochastic differential equations (written in Itô's form):

$$d\rho_t = 2\text{cotan}2\rho_t dt + dB_t^1,$$

$$d\theta_t = \frac{2}{\sin 2\rho_t}dB_t^2,$$

$$dz_t = \tan\rho_t dB_t^2.$$

This time, the operator

$$\frac{\partial^2}{\partial r^2} + 2\ cotan\ 2r\frac{\partial}{\partial r} + \left(\frac{2}{\sin 2r}\right)^2\frac{\partial^2}{\partial \theta^2}$$

corresponds to the Laplace-Beltrami operator on the sphere S^2 of radius $\frac{1}{2}$ in polar coordinates. The curve $(\rho(t), \theta(t))$ is a Brownian path on this sphere S^2 of radius $\frac{1}{2}$. The variation of the algebraic area $\mathcal{A}_t$ swept out by a curve $(\rho(t), \theta(t))$ in these coordinates is given by

$$d\mathcal{A}_t = \frac{(1 - \cos 2\rho(t))}{4}d\theta(t).$$

Therefore:

$$\begin{aligned} d\mathcal{A}_t &= \frac{(1-\cos 2\rho(t))}{4} d\theta(t) \\ &= \frac{(1-\cos 2\rho(t))}{2\sin 2\rho_t} dB_t^2 \\ &= \frac{2\sin^2\rho(t)}{4\sin\rho(t)\cos\rho(t)} dB_t^2 \\ &= \frac{dz_t}{2}. \end{aligned}$$

The z coordinate of our process is thus 2 times the area swept out by the Brownian motion on the sphere of dimension 2 *and radius* $\frac{1}{2}$.

The case of the $\mathbf{SL}(2,\mathbb{R})$ group
On $\mathbf{SL}(2,\mathbb{R})$, the cylindrical coordinates write

$$(r,\theta,z) \to \exp(r\cos\theta X + r\sin\theta Y)\exp(zZ)$$

$$= \begin{pmatrix} \cosh(r)\cos(z) + \sinh(r)\cos(\theta+z) & -\cosh(r)\sin(z) - \sinh(r)\sin(\theta+z) \\ \cosh(r)\sin(z) - \sinh(r)\sin(\theta+z) & \cosh(r)\cos(z) - \sinh(r)\cos(\theta+z) \end{pmatrix},$$

with

$$r>0,\ \theta\in[0,2\pi],\ z\in[-\pi,\pi].$$

Simple but tedious computations show that in these coordinates, the left-regular representation sends the matrices X, Y and Z to the left-invariant vector fields:

$$X = \cos(\theta+2z)\frac{\partial}{\partial r} - \sin(\theta+2z)\left(\tanh r\frac{\partial}{\partial z} + \left(\frac{1}{\tanh r} - \tanh r\right)\frac{\partial}{\partial\theta}\right),$$

$$Y = \sin(\theta+2z)\frac{\partial}{\partial r} + \cos(\theta+2z)\left(\tanh r\frac{\partial}{\partial z} + \left(\frac{1}{\tanh r} - \tanh r\right)\frac{\partial}{\partial\theta}\right),$$

$$Z = \frac{\partial}{\partial z}.$$

And the right-regular representation sends the matrices X, Y and Z to the right-invariant vecor fields:

$$\hat{X} = \cos(\theta)\frac{\partial}{\partial r} + \sin(\theta)\left(\tanh r\frac{\partial}{\partial z} - \left(\frac{1}{\tanh r} + \tanh r\right)\frac{\partial}{\partial\theta}\right),$$

$$\hat{Y} = \sin(\theta)\frac{\partial}{\partial r} - \cos(\theta)\left(\tanh r\frac{\partial}{\partial z} - \left(\frac{1}{\tanh r} + \tanh r\right)\frac{\partial}{\partial\theta}\right),$$

$$\hat{Z} = \frac{\partial}{\partial z} - 2\frac{\partial}{\partial\theta}.$$

We therefore obtain

$$\begin{aligned}
L &= X^2 + Y^2 \\
&= \frac{\partial^2}{\partial r^2} + 2\left(\frac{1}{\tanh r} + \tanh r\right)\frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial z^2} \\
&+ \left(\frac{1}{\tanh r} - \tanh r\right)^2 \frac{\partial^2}{\partial \theta^2} + 2\left(1 - \tanh^2 r\right)\frac{\partial^2}{\partial \theta \partial z} \\
&= \frac{\partial^2}{\partial r^2} + 2\coth 2r \frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial z^2} \\
&+ \left(\frac{2}{\sinh 2r}\right)^2 \frac{\partial^2}{\partial \theta^2} + 2\tanh r \left(\frac{2}{\sinh 2r}\right)\frac{\partial^2}{\partial \theta \partial z}.
\end{aligned}$$

The invariant and, in fact, also symmetric measure for L is then given (up to a constant) by

$$d\mu = \frac{\sinh 2r}{2} dr d\theta dz.$$

As before, the choice of the constant is made to obtain a good convergence towards the Lebesgue measure of $\mathbb{R}^3$ which is the invariant measure for the Heisenberg group (see section 4.7). Recall the group $\mathbf{SL}(2,\mathbb{R})$ is unimodular and note that the invariant measure μ coincides with the bi-invariant Haar measure of the group. Note also that L commutes with $\frac{\partial}{\partial \theta}$ and with $\frac{\partial}{\partial z}$.

Remark 2.5.3 (Probabilistic interpretation)**.** *The computation of the left-regular representation shows that if $(X_t)_{t\ge 0}$ is the Markov process that is the matrix-valued solution of the stochastic differential equation (written in Stratonovitch form)*

$$dX_t = X_t.(X \circ dB_t^1 + Y \circ dB_t^2), \quad X_0 = 0,$$

where $(B_t^1, B_t^2)_{t\ge 0}$ is a two-dimensional Brownian motion, then, in law,

$$X_t = \exp\left(\rho_t(X\cos\theta_t + Y\sin\theta_t)\right).\exp(z_t Z), \quad t \ge 0,$$

where $(\rho_t, \theta_t, z_t)_{t\ge 0}$ solve the following stochastic differential equations (written in Itô's form):

$$d\rho_t = 2\mathrm{cotanh} 2\rho_t dt + dB_t^1,$$

$$d\theta_t = \frac{2}{\sinh 2\rho_t} dB_t^2,$$

$$dz_t = \tanh \rho_t dB_t^2.$$

This time, the operator

$$\frac{\partial^2}{\partial r^2} + 2\coth 2r \frac{\partial}{\partial r} + \left(\frac{2}{\sinh 2r}\right)^2 \frac{\partial^2}{\partial \theta^2}$$

corresponds to the Laplace-Beltrami operator on the hyperbolic space H^2 of sectional curvature 2 in polar coordinates. The curve $(\rho(t), \theta(t))$ is a Brownian path on this space. The variation of the algebraic area $\mathcal{A}_t$ swept out by a curve $(\rho(t), \theta(t))$ in these coordinates is given by

$$d\mathcal{A}_t = \frac{(\cosh 2\rho(t) - 1)}{4} d\theta(t).$$

Therefore:

$$\begin{aligned} d\mathcal{A}_t &= \frac{(\cosh 2\rho(t)-1)}{4} d\theta(t) \\ &= \frac{(\cosh 2\rho(t)-1)}{2\sinh 2\rho_t} dB_t^2 \\ &= \frac{2\sinh^2\rho(t)}{4\sinh\rho(t)\cosh\rho(t)} dB_t^2 \\ &= \frac{dz_t}{2}. \end{aligned}$$

The z coordinate of our process is thus 2 times the area swept out by the Brownian motion on the hyperbolic space of dimension 2 *and sectional curvature* 2.

2.6 Symmetrical considerations

In this section, we will see some symmetrical properties that will be very useful in all the sequel. We begin by the θ invariance of the heat kernel.

The heat kernel on the three model spaces
We saw that in the three cases, the sublaplacian L commutes with $\frac{\partial}{\partial\theta}$ and with $\frac{\partial}{\partial z}$ From the commutation with $\Theta = \frac{\partial}{\partial\theta}$ which vanishes at the origin, we deduce that the heat kernel (issued from the identity) only depends on (r, z). It will then be denoted by $p_t(r, z)$ on the three spaces. Note also that sometimes the heat kernel on the Heisenberg group will also be denoted by $h_t(r, z)$. We hope there will be no confusions with the notation $p_t(g, g')$.
Let us sketch the proof of the above statement. For the moment, write the heat kernel as $p_t(r, \theta, z)$. The commtuation between L and Θ implies that the semigroup P_t commutes also with Θ. Then, as Θ vanishes at the identity, for all smooth compactly supported function f:

$$\begin{aligned} 0 = \Theta(P_t f)(0) &= P_t(\Theta f)(0) \\ &= \int_{\mathbb{G}} \Theta f(r,\theta,z) p_t(r,\theta,z) d\mu(r,\theta,z) \\ &= -\int_{\mathbb{G}} f(r,\theta,z) \Theta p_t(r,\theta,z) d\mu(r,\theta,z) \end{aligned}$$

since the adjoint of Θ with respect to the measure μ is $\Theta^* = -\Theta$. Since the last equality is valid for all smooth compactly supported function f, it implies $\Theta p_t(r, \theta, z) = 0$ which ends the proof.

The $\hat{\Gamma}$ of a radial function
We call a radial function (may be the term cylindrical function should be better) a function which does not depend on the variable θ in our cylindrical coordinates. For example, the heat kernel p_t is a radial function and therefore the results of this section apply to it.

Proposition 2.6.1. *Let f be a radial function on $\mathbb{G}$. Then,*

$$\hat{\Gamma}(f, f) = \Gamma(f, f)$$

where $\hat{\Gamma}(f, f) = (\hat{X} f)^2 + (\hat{Y} f)^2$.

Proof. Indeed, by a direct computation, one has:

$$\begin{array}{ll}\hat{\Gamma}(f,f) = \Gamma(f,f) - 4\Theta(f)Z(f) & \text{on } \mathbb{H},\\ \hat{\Gamma}(f,f) = \Gamma(f,f) - 4\Theta(f)Z(f) - 4(\Theta f)^2 & \text{on } \mathbf{SU}(2),\\ \hat{\Gamma}(f,f) = \Gamma(f,f) - 4\Theta(f)Z(f) + 4(\Theta f)^2 & \text{on } \mathbf{SL}(2,\mathbb{R}).\end{array}$$

□

Another symmetrical property

Here we shall describe a very elementary symmetrical property which explains what happens when we take a rotation of the vector fields X and Y.

Proposition 2.6.2. *Let $\alpha \in [0, 2\pi]$. Then:*

$$\begin{array}{ll}\cos\alpha X_{r,\theta,z} + \sin\alpha Y_{r,\theta,z} = X_{r,\theta-\alpha,z} & \textit{on } \mathbb{H},\\ \cos\alpha X_{r,\theta,z} + \sin\alpha Y_{r,\theta,z} = X_{r,\theta+\alpha,z} & \textit{on } \mathbf{SU}(2),\\ \cos\alpha X_{r,\theta,z} + \sin\alpha Y_{r,\theta,z} = X_{r,\theta-\alpha,z} & \textit{on } \mathbf{SL}(2,\mathbb{R}).\end{array}$$

Remark 2.6.3. *Of course, a similar proposition holds for the right invariant vector fields.*

2.7 The Γ_2 of a radial function

We saw before that we do not have any $CD(\rho,\infty)$ inequality for our sublaplacians L. However in this section we show that, on our three examples, Γ_2 of a radial function is non negative. This means the $CD(0,\infty)$ critrion is valid in restriction to radial functions.

Proposition 2.7.1 (The $\mathbb{H}$ case)**.** *Let f a smooth compactly supported function on $\mathbb{H}$ which only depends on (r,z), we have*

$$\Gamma(f,f) = \left(\frac{\partial f}{\partial r}\right)^2 + r^2\left(\frac{\partial f}{\partial z}\right)^2,$$

$$Lf = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + r^2\frac{\partial^2 f}{\partial z^2}$$

and

$$\Gamma_2(f,f) = \left(\frac{\partial^2 f}{\partial r^2}\right)^2 + \left(r^2\frac{\partial^2 f}{\partial z^2} - \frac{1}{r}\frac{\partial f}{\partial r}\right)^2 + 2\left(\frac{\partial f}{\partial z} + r\frac{\partial^2 f}{\partial r\partial z}\right)^2.$$

Proposition 2.7.2 (The $\mathbf{SU}(2)$ case)**.** *Let f a smooth compactly supported function on $\mathbf{SU}(2)$ which only depends on (r,z), we have*

$$\Gamma(f,f) = \left(\frac{\partial f}{\partial r}\right)^2 + \tan^2 r\left(\frac{\partial f}{\partial z}\right)^2,$$

$$Lf = \frac{\partial^2}{\partial r^2} + 2\ cotan\ 2r\frac{\partial}{\partial r} + \tan^2 r\frac{\partial^2}{\partial z^2}$$

and

$$\Gamma_2(f,f) = \left(\frac{\partial^2 f}{\partial r^2}\right)^2 + \left(\tan^2 r\frac{\partial^2 f}{\partial z^2} - \frac{2}{\sin 2r}\frac{\partial f}{\partial r}\right)^2 + 2\left(\frac{1}{\cos^2 r}\frac{\partial f}{\partial z} + \tan r\frac{\partial^2 f}{\partial r\partial z}\right)^2.$$

Proposition 2.7.3 (The $\mathbf{SL}(2,\mathbb{R})$ case)**.** *Let f a smooth compactly supported function on $\mathbf{SL}(2,\mathbb{R})$ which only depends on (r,z), we have*

$$\Gamma(f,f) = \left(\frac{\partial f}{\partial r}\right)^2 + \tanh^2 r \left(\frac{\partial f}{\partial z}\right)^2,$$

$$Lf = \frac{\partial^2}{\partial r^2} + 2\coth 2r \frac{\partial}{\partial r} + \tan^2 r \frac{\partial^2}{\partial z^2}$$

and

$$\Gamma_2(f,f) \quad = \quad \left(\frac{\partial^2 f}{\partial r^2}\right)^2 + \left(\frac{2}{\sinh 2r}\frac{\partial f}{\partial r} - \tanh^2 r \frac{\partial^2 f}{\partial z^2}\right)^2 + 2\left(\frac{1}{\cosh^2 r}\frac{\partial f}{\partial z} + \tanh r \frac{\partial^2 f}{\partial r \partial z}\right)^2.$$

Thus, on each of our three spaces, we obtain that for a smooth radial function f, $\Gamma_2(f,f) \geq 0$. This is an interesting fact which may be surprising if we think, for example, that this subelliptic $\mathbf{SL}(2,\mathbb{R})$ is the subelliptic model space with negative curvature.
The proof of these propositions is coming from simple but tedious computations. Note that the Γ_2 of a radial function corresponds in fact to the Γ_2 of the radial part of L since the coefficients in L do not depend on θ.

Chapter 3

CR manifolds

The goal of this chapter is clearly not to give a theoretical introduction to CR manifolds. The goal is rather to see that the subelliptic structures that we are dealing with, the Lie groups $\mathbf{SU}(2), \mathbb{H}$ and $\mathbf{SL}(2,\mathbb{R})$, belong in fact to a wider class of structures for which the methods of Chapter 5 should apply: namely, the CR manifolds of hypersurface type with a vanishing pseudo-Hermitian torsion of the Tanaka-Webster connexion. Indeed, these structures will satisfy some commutation and anti-symmetric conditions which are the one needed to make work the proof of the Li-Yau estimates of Chapter 5. Therefore one should be able to obtain a Li-Yau estimate of the form 5.1.10 in this larger setting.

We will also see that there exists a notion of curvature for this wider class and that the subelliptic structures on $\mathbf{SU}(2), \mathbb{H}$ and $\mathbf{SL}(2,\mathbb{R})$ have respectively a positive, a null and a negative constant curvature and therefore can be thought as some reference or model spaces in this wider class. Moreover, these structures admit a canonical sublaplacian which, as we will see, coïncide in our cases with the operator L.
A lot of the material presented here on CR manifolds is taken from the book [38].

3.1 A short account on CR manifolds

Here we begin with the theoretical definition of CR manifolds. The main interest of this definition will be to see that real submanifolds of $\mathbb{C}^n$ carry a natural structure of CR manifold.

Definition 3.1.1. *A CR manifold M is a $\mathcal{C}^\infty$ manifold endowed with a complex subbundle $T_{1,0}(M)$ of the complexified tangent bundle $T(M)\otimes\mathbb{C}$ satisfying*

$$T_{1,0}(M)\cap\overline{T_{1,0}(M)}=\{0\}$$

and the (formal) integrability condition

$$[\Gamma^\infty(T_{1,0}(M)),\Gamma^\infty(T_{1,0}(M))]\subset\Gamma^\infty(T_{1,0}(M)) \tag{3.1.1}$$

where $\Gamma^\infty(T_{1,0}(M))$ denotes the set of vector fields X such that $X_m\in T_{1,0;m}(M)$. If moreover M is of dimension m and $T_{1,0}(M)$ of dimension d, then the CR manifold M is said of type (d,k) where k is such that $2d+k=m$.

Let M be a CR manifold of type (d,k), we call the Levi distribution its real rank $2d$ subbundle $H(M)$ defined by

$$H(M) = Re\{T_{1,0}(M) \oplus \overline{T_{1,0}(M)}\}.$$

The map $J : H(M) \to H(M)$ given by

$$J\left(\frac{V+\overline{V}}{2}\right) = i\frac{V-\overline{V}}{2}$$

for any $V \in T_{1,0}(M)$ is a linear mapping of $H(M)$ such that $J^2 = -Id$. It endows therefore $H(M)$ with a complex structure.

The formal integrability condition (3.1.1) is equivalent to the two following conditions on the complex structure J:

$$[JX, Y] + [X, JY] \in H(M) \tag{3.1.2}$$

and

$$[JX, JY] - [X, Y] = J\left([JX, Y] + [X, JY]\right) \tag{3.1.3}$$

for all $X, Y \in H(M)$.
The proof of this result is easy since $V = X + iJX \in T_{1,0}(M)$ if $X \in H(M)$. Identically, $W = Y + iJY \in T_{1,0}(M)$ if $Y \in H(M)$. So

$$\begin{aligned} [V, W] &= [X + iJX, Y + iJY] \\ &= [X, Y] - [JX, JY] + i[JX, Y] + [X, JY] \end{aligned}$$

from which the equivalence easily follows.
The first and maybe most interesting examples of CR manifolds are the real submanifolds of some complex manifolds. For instance, a real hypersurface of $\mathbb{C}^{n+1}$ admit a natural CR structure of type $(n, 1)$ induced by the complex stucture of the ambient space

$$T_{1,0}(M) = T_{1,0}(\mathbb{C}^{n+1}) \cap T(M) \otimes \mathbb{C}.$$

In others words, $T_{1,0}(M)$ is the set of holomorphic tangent vectors to M. Therefore, the condition $T_{1,0}(M) \cap \overline{T_{1,0}(M)} = \{0\}$ and the integrability condition are clearly satified.
From now on, we are only interested in orientable CR manifold of real hypersurface type that is of type $(n, 1)$ for some $n \geq 1$. For these manifolds, it can be shown by some elementary considerations that there exist globally defined and nowhere vanishing differential forms θ such that $\theta_x(V) = 0$ for all points $x \in M$ and all tangent vectors $V \in H(M)_x$.
Such a differential form θ is called a pseudo-Hermitian structure on M.
Moreover all the other differential forms satisfying the same property write $\hat{\theta} = \lambda\theta$ where λ is a non vanishing smooth function on M and θ a pseudo-Hermitian structure. A pseudo-Hermitian structure θ enables to define the bilinear map g_θ

$$g_\theta(X, Y) = d\theta(X, JY)$$

for any $X, Y \in H(M)$. It can be shown this map is symmetric. If this map is non degenerate, it endows the subbundle $H(M)$ with a Lorentzian metric and if this map is positive definite, it endows the subbundle $H(M)$ with a metric. Note that the fact that this map is not degenerate does not depend on the choice of the pseudo-Hermitian structure θ and therefore is a CR invariant.

This explains the deep analogy between conformal geometry and CR manifolds, but here, this is not the point we are interested in. Of course, the positive definitness depends of this choice (take $\hat{\theta} = -\theta$).
Now if the CR manifold is non degenerate, one has the following result:

Proposition 3.1.2. *Let $(M, T_{1,0}(M))$ be a non-degenerate CR manifold and θ be a pseudo-Hermitian structure. Then there exists a nowhere vanishing tangent vector field T on M such that*

$$\theta(T) = 0, d\theta(T, .) = 0.$$

Moreover, the tangent bundle decomposes

$$T(M) = H(M) \oplus \mathbb{R}T.$$

This vector field T is called the characteristic direction or Reeb field of (M, θ).

For such a CR manifold there exists a canonical linear connection compatible with both the the complex structure J of the Levi distribution $H(M)$ and the metric g_θ.

Theorem 3.1.3. *Let $(M, T_{1,0}(M))$ be a non-degenerate CR manifold and θ be a pseudo-Hermitian structure. Let g_θ be the associated metric on $H(M)$. Let T be the Reeb field and J the complex structure on $H(M)$ (extended to $T(M)$ by setting $JT = 0$). There is a unique linear connection ∇ such that:*

- *The connection is horizontal, that is,*

  $$\nabla_X \Gamma^\infty(T(M)) \subset \Gamma^\infty(H(M))$$

  for any vector field X on M.
- $\nabla J = 0$
- $\nabla g_\theta = 0$
- *The torsion T_∇ of ∇ is pure, that is,*

  $$T_\nabla(X, Y) = d\theta(X, Y)T$$

  and

  $$T_\nabla(T, JX) = -JT_\nabla(T, X)$$

  for all $X, Y \in H(M)$.

Moreover the connection satisfies:

$$\nabla T = 0.$$

The torsion of a connection ∇ is defined by $T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. This connection ∇ is called the Tanaka-Webster connection. Note that the theorem 3.1.3 is not exactly written in the same way as Theorem 1.3 in [38], but the proof of their result can be easily adapted to get theorem 3.1.3.
In fact, we are only interested in a smaller class of CR manifolds: the ones for which the pseudo-Hermitian torsion of the Tanaka-Webster connection vanishes in the sense:

$$T_\nabla(T, X) = 0 \text{ for all } X \in H(M).$$

Moreover, we are only interested in the CR manifolds such that the metric g_θ is positive definite on $H(M)$.

Theorem 3.1.4. *Let $(M, T_{1,0}(M))$ be a non-degenerate CR manifold and θ be a pseudo-Hermitian structure. Let T be the Reeb field. Assume that the associated metric g_θ on $H(M)$ is positive definite. Let $X_1, \ldots, X_n$ be vector fields such that $(X_1(x), \ldots, X_n(x))$ is an orthonormal basis of $H(M)$ in each point $x \in U$ with U an open set of M. Assume also that the torsion of the Tanaka-Webster connection satisfies the vanishing condition:*

$$T_\nabla(T, X) = 0 \text{ for all } X \in H(M). \tag{3.1.4}$$

Then, the vector fields X_i for $i \leq 2n$ and the Reeb field T have to satisfy in each point of the open set U the following commutation relations:

$$[X_i, X_j] = \sum_{l=1}^{2n} \omega_{ij}^l X_l + \gamma_{ij} T \tag{3.1.5}$$

and

$$[X_i, T] = \sum_{l=1}^{2n} \delta_i^l X_l \tag{3.1.6}$$

for $1 \leq i, j \leq 2n$, where $\omega_{ij}^l, \gamma_{ij}$ and δ_i^l are smooth functions on U such that $\omega_{ij}^l = -\omega_{ji}^l$, $\gamma_{ij} = -\gamma_{ji}$ and

$$\delta_i^l = -\delta_l^i. \tag{3.1.7}$$

Reciprocally, if a manifold M admits (globally defined) horizontal vector fields $(X_i)_{1 \leq i \leq 2n}$ and a vertical Z which satisfy the above bracket relations (with Z instead of T of course). Let g be the metric defined by $g(X_i, X_j) = \delta_{i,j}$ and $g(Z, .) = 0$. Then there exists a unique linear connection ∇ such that:

- $\nabla g = 0$,
- $\nabla_{X_i} X_j$ *is horizontal for* $1 \leq i, j \leq n$,
- $\nabla Z = 0$,
- *if X, Y are horizontal vector fields, the torsion field $T_\nabla(X, Y)$ is vertical,*
- *if X is an horizontal vector field, the torsion field $T_\nabla(Z, X) = 0$.*

Moreover, in both cases, the connection is given by the formulas:

$$\nabla_{X_i} X_j = \sum_{k=1}^{2n} \frac{1}{2} \left(\omega_{ij}^k - \omega_{ik}^j - \omega_{jk}^i \right) X_k, \tag{3.1.8}$$

$$\nabla_T X_i = -\sum_{l=1}^{2n} \delta_i^l X_l \tag{3.1.9}$$

and

$$\nabla T = 0 \tag{3.1.10}$$

Remark 3.1.5. *By unicity of both the Tanaka-Webster connection and the connection ∇ of theorem 3.1.4, these two connections coïncide when they both are defined; that's why we denote both them in the same way. More precisely, if a manifold M is endowed with a CR structure (M, θ) of type $(n, 1)$ and if it admits vector fields $(X_i)_{1\leq i\leq 2n}$ and T which satisfy the above bracket relations and such that the vector fields X_i span the horizontal space $H(M)$ and form an orthonormal basis of the metric g_θ. Then the Tanaka-Webster connection has a vanishing pseudo-Hermitian torsion in the sense of equation (3.1.4) and coïncides with the connection ∇ and the Reeb vector field T write $T = \alpha Z$ for some constant $\alpha \in \mathbb{R}$.*
Reciprocally, if a manifold admits vector fields $(X_i)_{1\leq i\leq 2n}$ and T which satisfy the above bracket relations and a CR structure (M, θ) compatible in the sense that the X_i's are an orthonormal basis for g_θ. Then the connexion ∇ is the Tanaka-webster connection and hence satisfyies also $\nabla J = 0$ and $T_\nabla(T, JX) = -JT_\nabla(T, X)$ for all $X \in H(M)$.

Remark 3.1.6. *This theorem was generalized in [17] to the case of a codimension bigger than 1.*

Note that the conditions:

$$\begin{cases} \nabla g = 0 \\ \nabla_X \Gamma^\infty(T(M)) \subset \Gamma^\infty(H(M)) \end{cases}$$

and

$$\begin{cases} \nabla g = 0 \\ \nabla_X \Gamma^\infty(H(M)) \subset \Gamma^\infty(H(M)) \\ \nabla T = 0 \end{cases}$$

are equivalent.
Indeed, since $\nabla g = 0$, for all $X, Y, Z \in T(M)$,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

If one take $Z = T$ and use that $g(T, .) = 0$, one gets

$$g(Y, \nabla_X T) = 0.$$

And the equivalence is then clear.

Proof. Let ∇ be the Tanaka-Webwster connection of the manifold. Since the connection is horizontal and since is an orthonormal basis of $H(M)$, one has:

$$\nabla_{X_i} X_j = \sum_{k=1}^{2n} g(\nabla_{X_i} X_j, X_k) X_k.$$

Next, as $\nabla g = 0$, one has , for X, Y, Z vector fields on M,

$$X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{3.1.11}$$

Since the torsion is vertical, one can easily obtain from this that the Kozul identity holds for ∇, that is,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X.g(Y, Z) + Y.g(X, Z) - Z.g(X, Y) \\ &+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \end{aligned}$$

If we use the orthonormality asumptions on the X_i's and Z, we obtain:

$$\begin{aligned}\Gamma_{i,j}^k &= g(\nabla_{X_i}X_j, X_k)\\ &= \frac{1}{2}\left(g([X_i,X_j],X_k) - g([X_i,X_k],X_j) - g([X_j,X_k],X_i)\right)\\ &= \frac{1}{2}\left(\omega_{ij}^k - \omega_{ik}^j - \omega_{jk}^i\right)\end{aligned}$$

where $\Gamma_{i,j}^k$ denote the Christoffel symbols of the connection, that is $\nabla_{X_i}X_j = \sum_{k=1}^{2n}\Gamma_{i,j}^k X_k$. Similarly, one has

$$\nabla_T X_i = \sum_{l=1}^{2n} g(\nabla_T X_i, X_l)X_l.$$

The condition on the torsion $T_\nabla(T,X) = \nabla_T X - \nabla_X T - [T,X] = 0$ implies:

$$[X,T] \text{ is horizontal} \tag{3.1.12}$$

and

$$g(\nabla_T X_i, X_l) = -g([X_i,T],X_l) = -\delta_i^l. \tag{3.1.13}$$

By using the Kosul formula,

$$\begin{aligned}g(\nabla_T X_i, X_l) &= \frac{1}{2}\left(g([T,X_i],X_l) - g([T,X_l],X_i) - g([X_i,X_l],T)\right)\\ &= \frac{1}{2}\left(-\delta_i^l + \delta_l^i\right)\end{aligned}$$

This last equality together with (3.1.13) implies the condition:

$$\delta_i^l = -\delta_l^i$$

which ends the first part of the proof. For the reciprocal sense, the condition $\nabla g = 0$ and the fact that the torsion is vertical implies that if such connection exists the above Kozul formulas still holds. From the above computations and the bracket relations between the X_i's and T, if it exists the connection is given by the formulas (3.1.8), (3.1.9) and (3.1.10). Now what remains to do is to see, that the connection defined by these formulas (3.1.8), (3.1.9) and (3.1.10) satisfies the desired property. This is easily checked. Indeed, first we clearly have $\nabla_{X_i}X_j$ horizontal and $\nabla_Z = 0$. Next,

$$T_\nabla(X_i,X_j) = \nabla_{X_i}X_j - \nabla_{X_j}X_i - [X_i,X_j]$$

and

$$\begin{aligned}g(\nabla_{X_i}X_j - \nabla_{X_j}X_i - [X_i,X_j],X_k) &= \frac{1}{2}\left(\omega_{ij}^k - \omega_{ik}^j - \omega_{jk}^i\right) - \frac{1}{2}\left(\omega_{ji}^k - \omega_{jk}^i - \omega_{ik}^j\right) - \omega_{ij}^k\\ &= 0.\end{aligned}$$

Then the torsion field $T_\nabla(X_i,X_j)$ is vertical and satistisfies in fact

$$T_\nabla(X_i,X_j) = -\gamma_{ij}Z.$$

The pseudo-Hermitian torsion vanishes since:

$$\begin{aligned} T_\nabla(T, X_i) &= \nabla_T X_i - [T, X_i] \\ &= \sum_{l=1}^{2n} -\delta_i^l X_l + \sum_{l=1}^{2n} \delta_i^l X_l \\ &= 0. \end{aligned}$$

Finally,

$$0 = X_i g(X_j, X_k) \quad = \quad g(\nabla_{X_i} X_j, X_k) + g(X_j, \nabla_{X_i} X_k) + (\nabla_{X_i} g)(X_j, X_k),$$

but

$$\begin{aligned} g(\nabla_{X_i} X_j, X_k) + g(X_j, \nabla_{X_i} X_k) &= \frac{1}{2}\left(\omega_{ij}^k - \omega_{ik}^j - \omega_{jk}^i\right) + \frac{1}{2}\left(\omega_{ik}^j - \omega_{ij}^k - \omega_{kj}^i\right) \\ &= 0 \end{aligned}$$

so $\nabla_{X_i} g = 0$; and

$$0 = Zg(X_j, X_k) = g(\nabla_Z X_j, X_k) + g(X_j, \nabla_Z X_k) + (\nabla_Z g)(X_j, X_k)$$

but

$$\begin{aligned} g(\nabla_Z X_j, X_k) + g(X_j, \nabla_Z X_k) &= -\delta_j^k + \delta_k^j \\ &= 0 \end{aligned}$$

so $\nabla_Z g = 0$ and we are done. □

Now on a CR manifold (orientable and of hypersurface type) we can define a canonical differential operator: the sublaplacian.

Definition 3.1.7. *The sublaplacian is the differential operator L defined, for any smooth function u on M, by*

$$Lu = div(\nabla^H u) \tag{3.1.14}$$

where ∇_H is the horizontal gradient, that is,

$$du.X = X(u) = g_\theta(\nabla^H u, X)$$

for a vector field X and div is given by

$$div(X) = trace\{Y \in T(M) \to \nabla_Y X \in T(M)\}$$

for a vector field X.

Proposition 3.1.8. *Let $(X_i)_{i=1\ldots 2n}$ be an orthonormal basis of $H(M)$ for g_θ, then the sublaplacian L writes:*

$$L = \sum_{i=1}^{2n} X_i^2 - \nabla_{X_i} X_i. \tag{3.1.15}$$

Proof. By definition,

$$\begin{aligned}
Lu &= trace\{Y \in T(M) \to \nabla_Y \nabla^H u\} \\
&= trace\{X_i \to \nabla_{X_i} \nabla^H u\} \text{ since } \nabla(\Gamma^\infty(T(M)) \subset \Gamma^\infty(H(M)) \\
&= \sum_{i=1}^{2n} g_\theta(\nabla_{X_i}\nabla^H u, X_i) \text{ since } (X_i)_{i=1\ldots 2n} \text{ is an orthonormal basis for } g_\theta \\
&= \sum_{i=1}^{2n} X_i(g_\theta(\nabla^H u, X_i) - g_\theta(\nabla^H u, \nabla_{X_i} X_i) \text{ since } \nabla g_\theta = 0 \\
&= \sum_{i=1}^{2n} X_i(X_i u) - \nabla_{X_i} X_i u.
\end{aligned}$$

□

The canonical measure on the CR manifold is identified with the volume form $(d\theta)^n \wedge \theta$. In our setting, this volume form is just $dX \wedge dY \wedge dT$. The sublapacian L is then of course self-adjoint with respect to this measure and the adjoint of the vector field T is just $T^* = -T$.
Now we can show that, if the pseudo-Hermitian torsion of the Tanaka-Webster connection vanishes, the sublaplcian L satisfies the two conditions needed for the proof of the Li-Yau estimate.

Proposition 3.1.9. *Let $(M, T_{1,0}(M))$ be a non-degenerate CR manifold and θ a pseudo-Hermitian structure. Let T be the Reeb field. Assume that the associated metric g_θ on $H(M)$ is positive definite. Let $X_1, \ldots, X_n$ be vector fields such that $(X_1(x), \ldots, X_n(x))$ is an orthonormal basis of $H(M)$ in each point $x \in U$ with U an open set of M. Assume also that the pseudo-Hermitian torsion of the Tanaka-Webster connection vanishes. Then, one has:*

$$[L, T] = 0 \tag{3.1.16}$$

and

$$\sum_{i=1}^{2n} X_i(f)[X_i, T](f) = 0 \tag{3.1.17}$$

for all smooth function f.

Proof. For the first point, we begin by showing that $[L, T]$ is a vector field. Indeed,

$$\begin{aligned}
[L, T] &= \sum_{i=1}^{2n} [X_i^2, T] - \nabla_{X_i}[X_i, T] \\
&= \sum_{i=1}^{2n} X_i[X_i, T] + [X_i, T]X_i - \nabla_{X_i}[X_i, T] \\
&= \sum_i \sum_l \delta_i^l X_l X_i + X_i(\delta_i^l X_l) + [\omega_{il}^i X_l, T] \\
&= \sum_i \sum_l \left(X_i(\delta_i^l) X_l - T(\omega_{il}^i) X_l + \sum_k \omega_{il}^i \delta_l^k \right.
\end{aligned}$$

where we have used the crucial fact that $\delta_i^l = -\delta_l^i$. Now we show that $[L,T]^* = [L,T]$. Indeed, for f and g smooth functions:

$$\begin{aligned}
\langle [L,T]^* f, g\rangle &= \langle f, [L,T]g\rangle \\
&= \langle f, LTg\rangle - \langle f, TLg\rangle \\
&= \langle Lf, Tg\rangle + \langle Tf, Lg\rangle \\
&= -\langle TLf, g\rangle + \langle LTf, g\rangle \\
&= \langle [L,T]f, g\rangle
\end{aligned}$$

Eventually, a vector field V such that $V^* = V$ is identically zero. Indeed, for f and g smooth functions:

$$\begin{aligned}
0 = \langle V1, fg\rangle &= \langle V^*1, fg\rangle \\
&= \langle 1, V(fg)\rangle \\
&= \langle f, Vg\rangle + \langle g, Vf\rangle \\
&= 2\langle f, Vg\rangle.
\end{aligned}$$

As the above result is valid for any smooth functions f and g, it implies $V = 0$.
For the second point, one has

$$\begin{aligned}
\sum_i X_i(f)[X_i, Z](f) &= \sum_i \sum_l \delta_i^l X_i X_l \\
&= 0
\end{aligned}$$

where we used, as before, that $\delta_i^l = -\delta_l^i$. □

3.2 Our model spaces seen as CR manifolds

In this section, we will take a look at our three model spaces and see that they naturally carry a CR stucture. Recall that our model spaces are some Lie group $\mathbb{G}$ and that in each case a basis of the tangent space is given by the left-invariant vectors fields X, Y, Z which satisfy the relations:

$$[X,Y] = 2Z,$$

$$[X,Z] = -2\rho Y,$$

and

$$[Y,Z] = 2\rho X.$$

If we set $H(M) = \mathrm{Vect}(X,Y)$, consider J to be the linear endomorphism of $H(M)$ defined by

$$JX = Y \text{ and } JY = -X$$

and choose θ to be the differential form such that

$$\theta(X) = \theta(Y) = 0,\ d\theta(X,Y) = 1.$$

Then, we see that $J^2 = -Id$ and that the conditions 3.1.2 and 3.1.3. For the conditions 3.1.2 and 3.1.3, note that

$$[JA, B] + [A, JB] = 0$$

and

$$[JA, JB] - [A, B] = 0$$

for all $A, B \in \mathrm{Vect}(X, Y)$. Therefore, it endows our Lie group $(\mathbb{G}, \theta)$ with an abstract CR structure. To exploit the computations we have done, we set $X_1 = X$ and $X_2 = Y$ and we have

$$w_{12} = 0,\ \gamma_{12} = 2,\ \delta_1^1 = \delta_2^2 = 0,\ \delta_1^2 = -2\rho, \delta_2^1 = 2\rho.$$

We then see that we fall in the previous setting. The Tanaka-Webster connection is then given by:

$$\nabla_X X = \nabla_X Y = \nabla_Y X = \nabla_Y Y = 0$$

and

$$\nabla_Z X = 2\rho Y,\ \nabla_Z Y = -2\rho X$$

and has a vanishing pseudo-Hermitian torsion. The Reeb field is given by $T = -2Z$. and the conditions on θ and J implies that (X, Y) is an orthonormal basis for g_θ of $H(M)$. The sublaplacian then just writes:

$$L = X^2 + Y^2.$$

Of course, we can also look at the models spaces as real submanifolds of $\mathbb{C}^2$.
In this spirit, we can identify the Heisenberg group with the boundary of the Siegel domain:

$$\Omega = \{(z, w) \in C \times C, Im(w) > \frac{|z|^2}{2}\}$$

by the map:

$$f : \mathbb{H} \to \partial\Omega,\ f(x, y, z) = ((a_1, b_1), (a_2, b_2)) = (y + ix, z + \frac{i}{2}(x^2 + y^2)).$$

The change between x and y is only there to obtain a full consistant CR isomoprhism. This map is a CR isomorphism between the Heisenberg group in exponential coordinates onto $\partial\Omega$ where the CR structure on $\partial\Omega$ is the canonical CR structure obtained as an hypersurface of $\mathbb{C}^2$. Let us explicit the canonical structure of $\partial\Omega$. The implicit equation which defines $\partial\Omega$ is:

$$Im(z_2) = \frac{1}{2}|z_1|^2$$

that is

$$\frac{z_1 \bar{z}_1}{2} - \frac{z_2 - \bar{z}_2}{2i} = 0.$$

The holomorphic tangent vector fields $T_{1,0}(\partial\Omega)$ are spanned by the vector field

$$V_{\mathbb{H}} = 2i(\partial_{z_1} + i\bar{z}_1 \partial_{z_2}).$$

This vector field $V_{\mathbb{H}}$ can be written:

$$V_{\mathbb{H}} = A_{\mathbb{H}} + iB_{\mathbb{H}}$$

with $A_{\mathbb{H}}$ and $B_{\mathbb{H}}$ the real vector fields

$$A_{\mathbb{H}} = +\partial_{y_1} - x_1 \partial_{x_2} + y_1 \partial_{y_2}.$$

and

$$B_{\mathbb{H}} = \partial_{x_1} + y_1\partial_{x_2} + x_1\partial_{y_2}.$$

To see the CR isomorphism, note that the differential of f is given by:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ x & y & 0 \end{pmatrix}$$

and the images of $X = (\partial_x - y\partial_z)$ and $Y = \partial_y + x\partial_z)$ by df are:

$$df.X = \partial_{x_1} - y_1\partial_{x_2} + x_1\partial_{y_2} = \partial_{b_1} - a_1\partial_{a_2} + b_1\partial_{b_2}$$

and

$$df.Y = \partial_{y_1} + x_1\partial_{x_2} + y_1\partial_{y_2} = \partial_{b_1} + b_1\partial_{a_2} + a_1\partial_{y_2}$$

which coïncide respectively with $A_{\mathbb{H}}$ and $B_{\mathbb{H}}$.
For the group $\mathbf{SU}(2)$, we can do the same by identifying it with the unit sphere S^3 in $\mathbb{C}^2$. Let us do the identifaction by

$$\begin{pmatrix} z_1 & \bar{z_2} \\ -z_2 & \bar{z_1} \end{pmatrix} \in \mathbf{SU}(2) \to (z_1, z_2) \in S^3.$$

The equation for the sphere is:

$$|z_1|^2 + |z_2|^2 = 1.$$

Let us explicit the canonical CR structure of the sphere. The complex holomorphic tangent vectors to S^3, $T_{1,0}(S^3)$, are spanned by the vector:

$$V_{\mathbf{SU}(2)} = 2(-\bar{z_2}\partial_{z_1} + \bar{z_1}\partial_{z_2});$$

which can be written:

$$V_{\mathbf{SU}(2)} = A_{\mathbf{SU}(2)} + iB_{\mathbf{SU}(2)}$$

with $A_{\mathbf{SU}(2)}$ and $B_{\mathbf{SU}(2)}$ the real vector fields

$$A_{\mathbf{SU}(2)} = -x_2\partial_{x_1} + y_2\partial_{y_1} + x_1\partial_{x_2} - y_1\partial_{y_2}$$

and

$$B_{\mathbf{SU}(2)} = y_2\partial_{x_1} + x_2\partial_{y_1} - y_1\partial_{x_2} - x_1\partial_{y_2}.$$

We then see that our identification between $\mathbf{SU}(2)$ and S^3 in $\mathbb{C}^2$ is a CR isomorphism, since $A_{\mathbf{SU}(2)}$ and $B_{\mathbf{SU}(2)}$ coïncide respectively with the vector fields X and Y on $\mathbf{SU}(2)$ in this choice of coordinates.
For the group $\mathbf{SL}(2,\mathbb{R})$, we embed it in $\mathbb{C}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R}) \to (a + ib, c + id) \in \mathbb{C}^2.$$

Let us call M the image in $\mathbb{C}^2$. M is the submanifold of $\mathbb{C}^2$ defined by the equation

$$\frac{(z_1 + \bar{z_1})}{2}\frac{(z_2 - \bar{z_2})}{2i} - \frac{(z_1 - \bar{z_1})}{2i}\frac{(z_2 + \bar{z_2})}{2} = 1$$

that is by the equation:

$$-z_1\bar{z}_2 + \bar{z}_1 z_2 = 2i.$$

The CR structure on M is then given by: $T_{1,0}(M) = \mathrm{Vect} V_{\mathbf{SL}(2,\mathbb{R})}$ where

$$V_{\mathbf{SL}(2,\mathbb{R})} = 2(\bar{z}_1\partial_{z_1} + \bar{z}_2\partial_{z_2});$$

which can be written:

$$V_{\mathbf{SL}(2,\mathbb{R})} = A_{\mathbf{SL}(2,\mathbb{R})} + iB_{\mathbf{SL}(2,\mathbb{R})}$$

with $A_{\mathbf{SL}(2,\mathbb{R})}$ and $B_{\mathbf{SL}(2,\mathbb{R})}$ the real vector fields

$$A_{\mathbf{SL}(2,\mathbb{R})} = x_1\partial_{x_1} - y_1\partial_{y_1} + x_2\partial_{x_2} - y_2\partial_{y_2}$$

and

$$B_{\mathbf{SL}(2,\mathbb{R})} = -(y_1\partial_{x_1} + x_1\partial_{y_1} + y_2\partial_{x_2} + x_2\partial_{y_2}).$$

We then see that this identification is a CR isomorphism, since $A_{\mathbf{SL}(2,\mathbb{R})}$ and $B_{\mathbf{SL}(2,\mathbb{R})}$ coïncide respectively with the vector fields X and Y in this choice of coordinates on $\mathbf{SL}(2,\mathbb{R})$.

3.3 The curvature of our CR manifolds

The pseudo Hermitian curvature is given, as in the Riemannian case, by the tensor

$$R(U,V)W = -\left(\nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W\right).$$

Its sectional curvature is given by:

$$Sec(U,V) = g(R(U,V)U,V).$$

In our case, we only have to compute the sectional curvature of X and Y. First,

$$\begin{aligned} R(X,Y)X &= \nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X \\ &= 2\nabla_Z X \\ &= 4\rho Y, \end{aligned}$$

then the sectional curvature which is given by $Sec(X,Y) = g(R(X,Y)X,Y)$ has value 4ρ. Since we are only interested in the curvature in the plane X,Y, this sectional curvature can also be seen as a Ricci curvature for the CR manifold. This explains why we ce can consider the subelliptic structures on $\mathbf{SU}(2), \mathbb{H}$ and $\mathbf{SL}(2,\mathbb{R})$ as some model spaces of subelliptic geometry.

3.4 The Cayley transform

In this section, we show that our three model are conformally equivalent. This fact is well known between $\mathbb{H}$ and $\mathbf{SU}(2)$ (see [62] for example) and was used by Jerison and Lee [58] to obtain the optimal contants in the Sobolev inequality on $\mathbb{H}$ and $\mathbf{SU}(2)$. A similar work should be possible for between $\mathbf{SL}(2,\mathbb{R})$ and $\mathbb{H}$.
The Cayley transform is the restriction to $S^3 - \{-e_3\}$, $e_3 = (0,0,1,0)$ of the mapping

$$\Phi : (\omega_1,\omega_2) \in \mathbb{C}^2 - \{\omega_1 = 1\} \to (z_1,z_2) = \left(\frac{\omega_1}{1+\omega_2}, \frac{i}{2}\frac{1-\omega_2}{1+\omega_2}\right).$$

The image of $S^3 - \{-e_3\}$ is exactly $\partial\Omega$ the boundary of the Siegel domain Ω. Indeed

$$\begin{aligned}\frac{1-\omega_2}{1+\omega_2} &= \frac{(1-\omega_2)(1+\bar{\omega_2})}{|1+\omega_2|^2}\\ &= \frac{1-|\omega_2|^2-2iIm\omega_2}{|1+\omega_2|^2},\end{aligned}$$

so

$$Im\left(\frac{i}{2}\frac{1-\omega_2}{1+\omega_2}\right)=\frac{1}{2}\frac{|\omega_1|^2}{|1+\omega_2|^2}.$$

The map Φ is holomorphic in ω_1 and ω_2 and its differential writes:

$$d\Phi=\begin{pmatrix}\frac{1}{1+\omega_2} & -\frac{\omega_1}{(1+\omega_2)^2}\\ 0 & -\frac{i}{(1+\omega_2)^2}\end{pmatrix}$$

and it maps the vector $V_{\mathbf{SU}(2)}=2(-\bar{\omega_2}\partial_{\omega_1}+\bar{\omega_1}\partial_{\omega_2})$ on

$$\begin{aligned}d\Phi.V_{\mathbf{SU}(2)} &= -\frac{2(1+\bar{\omega_2})}{(1+\omega_2)^2}\left(\partial_{z_1}+\frac{i\bar{\omega_1}}{1+\bar{\omega_2}}\partial_{z_2}\right)\\ &= i\frac{(1+\bar{\omega_2})}{(1+\omega_2)^2}V_{\mathbb{H},\Phi(\omega_1,\omega_2)}.\end{aligned}$$

There exist also a conformal map between the $\mathbf{SL}(2,\mathbb{R})$ group and the boundary of the Siegel domain. This map is given by

$$\psi:(\omega_1,\omega_2)\to(z_1,z_2)=\left(\frac{1}{\omega_1+\omega_2},\frac{1}{4}\frac{\omega_1-\omega_2}{\omega_1+\omega_2}\right).$$

and its differential by

$$d\psi=\frac{1}{(\omega_1+\omega_2)^2}\begin{pmatrix}1 & 1\\ \frac{\omega_2}{2} & \frac{-\omega_1}{2}\end{pmatrix};$$

thus, recalling $-\omega_1\bar{\omega_2}+\omega_2\bar{\omega_1}=2i$, we see it maps the vectors fields $V_{\mathbf{SL}(2,\mathbb{R})}=2(\bar{\omega_1}\partial_{\omega_1}+\bar{\omega_2}\partial_{\omega_2})$

$$\begin{aligned}d\psi.V_{\mathbf{SL}(2,\mathbb{R})} &= \frac{2(\bar{\omega}_1+\bar{\omega_2})}{(\omega_1+\omega_2)^2}\left(\partial_{z_1}+\frac{i}{\bar{\omega_1}+\bar{\omega_2}}\partial_{z_2}\right)\\ &= -i\frac{(\bar{\omega_1}+\bar{\omega_2})}{(\omega_1+\omega_2)^2}V_{\mathbb{H}}.\end{aligned}$$

Chapter 4

Spectral and Integral representations of the heat kernels

In this section, we will derive some explicit representations of the heat kernels. Of course the expression of the heat kernel on the Heisenberg is well known. This expression is called Gaveau's formula and is due to both Gaveau [48] and Hulanicki [57] and in fact also appeared in the work on Lévy on the area swept out by a Brownian motion in $\mathbb{R}^2$ [69]. Here we will present a proof of this result due to Faraut (see [43]). This proof is based on the spherical Fourier transform on the Heisenberg group and uses the generating series of the Laguerre polynomials. For the $\mathbf{SU}(2)$ group, by taking the classical Fourier series in the z variable in our cylindrical coordinates, we obtain a spectral decomposition of the heat kernel with some Jacobi polynomials. Unfortunately, we do not manage to extend this method in the $\mathbf{SL}(2,\mathbb{R})$ case.
We also develop another approach to obtain a different representation of the heat kernel. This approach works both for $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$. This method is based on the link between the sublaplacian L and the Laplace Beltrami operator Δ on S^3 for the $\mathbf{SU}(2)$ group and between the sublaplacian L and the Casimir operator $\square$ for the group $\mathbf{SL}(2,\mathbb{R})$. This enables us to obtain integral representations for the heat kernels in which appear the classical heat kernel on the 3-sphere for $\mathbf{SU}(2)$ and the classical heat kernel on the 3-dimensional hyperbolic space for $\mathbf{SL}(2,\mathbb{R})$. This integral are somehow similar to the Gaveau formula for the Heisenberg group and we can extend some analytic technics of [19] to obtain asymptotics of the heat kernels in small times and as a consequence expressions of the subriemannian distance. These explicit integral representations also enables us to obtain a convergence result for $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ at the level of the diffusion. We hope that, in the future, one should be able to obtain optimal estimates for the heat kernel on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ by making a careful study of these explicit representations as it is the case for the Heisenberg group.
The spectral decomposition of the heat kernel on $\mathbf{SU}(2)$ was also done in [18]. Some spectral decompositions on $\mathbb{H}, \mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ were also obtained in [1].

4.1 The heat kernel on the Heisenberg group $\mathbb{H}$

In this section we set the Gaveau's formula for the heat kernel on the Heisenberg group.

Proposition 4.1.1. *With respect to the Lebesgue measure $rdrd\theta dz$ the heat kernel associated to* *the semigroup* $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ *writes*

$$h_t(r,z) = \frac{1}{16\pi^2 t^2}\int_{-\infty}^{+\infty} e^{\frac{i\lambda z}{2t}} e^{-\frac{r^2}{4t}\lambda\coth\lambda}\frac{\lambda}{\sinh\lambda}d\lambda. \tag{4.1.1}$$

Here, for the sake of completness, we give a proof of this formula. This proof uses the spherical Fourier transform on the Heisenberg group. This proof appears in the book of Faraut [43] and we refer to this book for all the properties of the spherical Fourier transform on the Heisenberg group.

Proof. We work in the case of the Heisenberg group of dimension 3, but the proof extends to the Heisenberg groups of higger dimensions. We consider the following Cauchy problem:

$$\begin{cases} \partial_t u = Lu \\ u(x,y,z;0) = f(x,y,z) \end{cases}$$

for f a continuous function on $\mathbb{H}$. If we assume that f and u are radial function, then taking the spherical Fourier transform, we get:

$$\begin{cases} \partial_t \hat{u}(\lambda,m;t) = -4|\lambda|\left(m+\frac{1}{2}\right)\hat{u}(\lambda,m;t) \\ \hat{u}(\lambda,m;0) = \hat{f}(\lambda,m) \end{cases}$$

for $\lambda\in\mathbb{R}$, $m\in\mathbb{N}$ and $t>0$
and thus,

$$\hat{u}(\lambda,m;t) = e^{-4|\lambda|\left(m+\frac{1}{2}\right)t}\hat{f}(\lambda,m).$$

Hence, if we define p_t to be the radial function on $\mathbb{H}$ such that

$$\hat{p}_t(\lambda,m) = e^{-4|\lambda|\left(m+\frac{1}{2}\right)t};$$

the solution of the Cauchy problem is then given by

$$u(x,y,z;t) = (f*p_t)(x,y,z)$$

where $f*g$ denotes the convolution in the Heisenberg group defined by

$$f*g(x,y,z) = \int_{\mathbb{H}} f(x',y',z')\, g\left((-x',-y',-z').(x,y,z)\right) dx'dy'dz'.$$

The Fourier inversion formula leads us to set:

$$p_t(x,y,z) = \frac{1}{2\pi^2}\int_{\lambda=-\infty}^{\infty}\sum_{m\geq 0} e^{-4|\lambda|\left(m+\frac{1}{2}\right)t}\,\omega_{\lambda,m}\,|\lambda|d\lambda$$

with

$$\omega_{\lambda,m} = e^{i\lambda z}\, e^{-\frac{1}{2}|\lambda|(x^2+y^2)}\, L_m(|\lambda|(x^2+y^2))$$

and where L_m is the Laguerre polynomial defined by

$$L_m(x) = \frac{1}{m!}e^x\left(\frac{d}{dx}\right)^m\left(e^{-x}x^m\right).$$

The Laguerre polynomials satsify the following generating series:

$$\sum_{m\geq 0} L_m(x)\, t^m = \frac{1}{1-t} e^{-\frac{t}{1-t}x}.$$

Therefore:

$$\begin{aligned} p_t(x,y,z) &= \frac{1}{2\pi^2}\int_{\lambda=-\infty}^{\infty} e^{i\lambda z} \frac{e^{-2|\lambda|t}}{1-e^{-4|\lambda|t}} \exp\left[-\left(\frac{1}{2}+\frac{e^{-4|\lambda|t}}{1-e^{-4|\lambda|t}}\right)\right] |\lambda| d\lambda \\ &= \frac{1}{4\pi^2}\int_{\lambda=-\infty}^{\infty} e^{i\lambda z} \frac{|\lambda|}{\sinh 2|\lambda|t} \exp\left[-\frac{1}{2}|\lambda|(x^2+y^2)\coth 2|\lambda|t\right] d\lambda. \end{aligned}$$

The Gaveau formula is then just obtained by an easy change of variables. □

4.2 Spectral decomposition of the heat kernel on $\mathbf{SU}(2)$

In this section we will derive a spectral decomposition for the heat kernel on the space $\mathbf{SU}(2)$. This result is linked with the Riemannian submersion described in Section 2.3.

Proposition 4.2.1. *With respect to the measure $d\mu = \frac{\sin 2r}{2} dr d\theta dz$, the heat kernel associated to the semigroup $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ on $\mathbf{SU}(2)$ writes, for $t>0$, $r\in[0,\pi/2)$, $z\in[-\pi,\pi]$,*

$$p_t(r,z) = \frac{1}{2\pi^2}\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}(2k+\mid n\mid+1)e^{-(4k(k+|n|+1)+2|n|)t}e^{inz}(\cos r)^{|n|}P_k^{0,|n|}(\cos 2r),$$

where

$$P_k^{0,|n|}(x) = \frac{(-1)^k}{2^k k!(1+x)^{|n|}}\frac{d^k}{dx^k}\left((1+x)^{|n|}(1-x^2)^k\right)$$

is a Jacobi polynomial.

Proof. Since the points (r,θ,z) and $(r,\theta,z+2\pi)$ are the same, we can define $p_t(r,z)$ for all $z\in\mathbb{R}$ and it is 2π-periodic. The idea is then to expand $p_t(r,z)$ as a Fourier series in z:

$$p_t(r,z) = \sum_{n=-\infty}^{+\infty} e^{inz}\Phi_n(t,r)$$

Since $p_t(r,z)$ satisfies the partial differential equation,

$$\frac{\partial p_t}{\partial t} = Lp_t,$$

we obtain for Φ_n the following equation

$$\frac{\partial \Phi_n}{\partial t} = \frac{\partial^2\Phi_n}{\partial r^2} + 2\ \text{cotan}\ 2r\frac{\partial\Phi_n}{\partial r} - n^2\tan^2 r\Phi_n.$$

We recognize that it is the equation for the harmonic oscillator on the sphere of dimension 2 and radius $\frac{1}{2}$ If we look for a solution under the form

$$\Phi_n(t,r) = e^{-2nt}(\cos r)^{|n|}f_n(t,r)$$

then f_n satisfies the following equation:

$$\frac{\partial f_n}{\partial t} = \frac{\partial^2 f_n}{\partial r^2} + \left(\frac{1}{\tan r} - (2n+1)\tan r \right) \frac{\partial f_n}{\partial r}.$$

Actually, the change of functions: $f_n(t,r) = g_n(t, \cos(2r))$ gives that for all $x \in [-1,1]$, one has:

$$\frac{\partial g_n}{\partial t}(x) = 4\mathcal{G}_n(g_n)(x)$$

where

$$\mathcal{G}_n = (1-x^2)\frac{\partial^2}{\partial x^2} + (|n| - (2+|n|)x)\frac{\partial}{\partial x}.$$

It is well-known that eigenvectors of $\mathcal{G}_n$ are the Jacobi polynomials:

$$P_k^{0,|n|}(x) = \frac{(-1)^k}{2^k k!(1+x)^{|n|}} \frac{d^k}{dx^k} \left((1+x)^{|n|}(1-x^2)^k \right)$$

In fact we have

$$\mathcal{G}_n(P_k^{0,|n|})(x) = -k(k+n+1)P_k^{0,|n|}(x)$$

So we are finally led to put

$$p_t(r,z) = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \alpha_{k,n} e^{-(4k(k+|n|+1)+2|n|)t} e^{inz} (\cos r)^{|n|} P_k^{0,|n|}(\cos 2r)$$

for some $\alpha_{k,n}$. It is clear that p_t satisfies the partial differential equation $\frac{\partial p_t}{\partial t} = Lp_t$. Now we have to determine the $\alpha_{k,n}$. They will be determined by the initial condition at time 0.
We have to check that

$$2\pi \int_0^{\frac{\pi}{2}} \int_0^{2\pi} p_t(r,z) f(r,z) \frac{\sin(2r)}{2} drdz \to_{t\to 0} f(0,0)$$

for every smooth function f. Indeed

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\pi}^{\pi} p_t(r,z) f(r,\theta,z) \frac{\sin(2r)}{2} drd\theta dz = 2\pi \int_0^{\frac{\pi}{2}} \int_0^{2\pi} p_t(r,z)\tilde{f}(r,z)\frac{\sin(2r)}{2} drdz$$

with

$$\tilde{f}(r,z) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta,z) d\theta.$$

Moreover, it is sufficient to do it for the f of the form $f(r,z) = g(\cos(2r))h(z)$. Now to do this, we will use the orthogonal properties of the Jacobi polynomials. We will use the fact that for all $n \in \mathbb{N}$, $(P_k^{0,|n|})_{k\geq 0}$ is an orthogonal basis of $L^2([-1,1], (1+x)^{|n|}du)$ with $||P_k^{0,|n|}||^2 = \frac{2^{|n|+1}}{2k+|n|+1}$.

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} p_t(r,z) f(r,z) \sin(2r) drdz \\
&\to_{t\to 0} \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \alpha_{k,n} e^{inz} (\cos r)^{|n|} P_k^{0,|n|}(\cos 2r) g(\cos(2r)) h(z) \sin(2r) drdz \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left(\sum_{k=0}^{+\infty} \alpha_{k,n} \int_0^{\frac{\pi}{2}} (\cos r)^{|n|} P_k^{0,|n|}(\cos 2r) g(\cos(2r)) \sin(2r) dr \right) \left(\int_0^{2\pi} h(z) e^{inz} dz \right).
\end{aligned}$$

And we note that we are done if

$$\left(\sum_{k=0}^{+\infty}\alpha_{k,n}\int_0^{\frac{\pi}{2}}(\cos r)^{|n|}P_k^{0,|n|}(\cos 2r)g(\cos(2r))\sin(2r)dr\right)=\frac{g(1)}{2^p i^2}.$$

Indeed, multiplying by $2\pi^2$, we then obtain

$$g(1)\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\int_0^{2\pi}h(z)e^{inz}dz\right)=g(1)h(0)=f(0,0)$$

and we recognize the Fourier serie of h taken at the point 0. The goal is now to find some $\alpha_{k,n}$ so that the preceding result is true. The change of variable $u=\cos(2r)$ gives

$$\begin{aligned}&\sum_{k=0}^{+\infty}\alpha_{k,n}\int_0^{\frac{\pi}{2}}(\cos r)^{|n|}P_k^{0,|n|}(\cos 2r)g(\cos(2r))\sin(2r)dr\\ =&\sum_{k=0}^{+\infty}\frac{\alpha_{k,n}}{2}\int_{-1}^{1}g(u)\left(\frac{1+u}{2}\right)^{|n|/2}P_k^{0,|n|}(u)du\end{aligned}$$

But, as we said it before, $(P_k^{0,|n|})_{k\geq 0}$ is an orthogonal basis of $L^2([-1,1],(1+u)^{|n|}du)$ and it satisfies: $||P_k^{0,|n|}||^2=\frac{2^{|n|+1}}{2k+|n|+1}$ and $P_k^{0,|n|}(1)=1$. So, if $\phi\in L^2([-1,1],(1+u)^{|n|}du)$, we have in the L^2 sense

$$\phi(x)=\sum_{k=0}^{+\infty}\frac{2k+|n|+1}{2^{|n|+1}}\left(\int_{-1}^{1}\phi(u)P_k^{0,|n|}(u)(1+u)^{|n|}du\right)P_k^{0,|n|}(x)$$

This equality is also valid pointwise if ϕ is smooth. By taking $\phi(u)=\frac{g(u)}{(\frac{1+u}{2})^{|n|/2}}$, we get

$$\begin{aligned}g(1)&=\sum_{k=0}^{+\infty}\frac{2k+|n|+1}{2^{|n|+1}}\int_{-1}^{1}\frac{g(u)}{(\frac{1+u}{2})^{|n|/2}}P_k^{0,|n|}(u)(1+u)^{|n|}du\\ &=\sum_{k=0}^{+\infty}\frac{2k+|n|+1}{2}\int_{-1}^{1}g(u)P_k^{0,|n|}(u)\left(\frac{1+u}{2}\right)^{|n|/2}du\end{aligned}$$

Therefore, to obtain the convergence when t goes to 0, we have to choose $\alpha_{k,n}=\frac{2k+|n|+1}{2\pi^2}$. □

Remark 4.2.2. *By using the representation theory of* $\mathbf{SU}(2)$*, a similar spectral decomposition is given in [18]. Nevertheless, for the sake of completeness, we included this elementary proof.*

4.3 Towards a spectral decomposition of the heat kernel on $\mathbf{SL}(2,\mathbb{R})$

In this section we ask if we can obtain by the same way as in the previous section a spectral decomposition for the subelliptic heat kernel on $\mathbf{SL}(2,\mathbb{R})$. Indeed, the points (r,θ,z) and $(r,\theta,z+2\pi)$ are also the same in this case and therefore we can define $p_t(r,z)$ for all $z\in\mathbb{R}$ by 2π-periodicity in the z variable.

As in the previous section, we can then expand the heat kernel $p_t(r,z)$ as a Fourier series in z:

$$p_t(r,z) = \sum_{n=-\infty}^{+\infty} e^{inz}\Phi_n(t,r).$$

and will try to do some similar changes of functions. Since $p_t(r,z)$ satisfies the partial differential equation,

$$\frac{\partial p_t}{\partial t} = Lp_t,$$

As the heat kernel does not depend on the variable θ, we are only concerned by the radial part of the Laplacian L

$$L_{rad} = \frac{\partial^2}{\partial r^2} + 2\coth 2r \frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial z^2}.$$

Since $p_t(r,z)$ satisfies the partial differential equation,

$$\frac{\partial p_t}{\partial t} = Lp_t,$$

we obtain for Φ_n the following equation

$$\frac{\partial \Phi_n}{\partial t} = \frac{\partial^2 \Phi_n}{\partial r^2} + 2\coth 2r \frac{\partial \Phi_n}{\partial r} - n^2 \tanh^2 r \Phi_n.$$

The change of variable similar to the one for $\mathbf{SU}(2)$

$$\Phi_n(t,r) = e^{-2|n|t} \frac{1}{(\cosh r)^{|n|}} f_n(t,r)$$

gives us

$$\frac{\partial^2 f_n}{\partial r^2} + [\coth r + (-2|n|+1)\tanh r]\frac{\partial f_n}{\partial r} = \frac{\partial f_n}{\partial t}.$$

Now by setting $f_n(t,r) = g_n(t, \cosh(2r))$, we have, for all $x \geq 1$:

$$\frac{\partial g_n}{\partial t}(x) = 4\mathcal{H}_n(g_n)(x)$$

where

$$\mathcal{H}_n = (x^2-1)\frac{\partial^2}{\partial x^2} + ((2+|n|)x - |n|)\frac{\partial}{\partial x}.$$

This partial differential equation is very similar to the one obtained on $\mathbf{SU}(2)$, one can note that $\mathcal{H}_n = -\mathcal{G}_n$. But this time as we are working on $[1,+\infty)$ which is not anymore a compact space, the spectral study of the equation is very much harder. It uses Jacobi functions of the first and the second kind. For the moment, by this method, we do not manage to obtain a spectral decomposition for the subelliptic heat kernel on $\mathbf{SL}(2,\mathbb{R})$.

4.4 An integral representation for the heat kernel on $\mathbf{SU}(2)$

In this section, we will also use the Riemannian submersion that we introduce in section 2.3. But this time we will use the link between the subriemanian metric and the Riemannian one on the same 3 dimensional sphere, or more precisely between the Laplace-Beltrami operator Δ on the 3 dimensional sphere and the sublaplacian L. Recall that with our choices of vector fields, these operators write:

$$L = X^2 + Y^2$$

and

$$\Delta = X^2 + Y^2 + Z^2.$$

Recall also

$$[L, Z] = 0.$$

Therefore one can write the following formula which linked the two semigroups associated with L and with Δ:

$$e^{tL} = e^{-tZ^2} e^{t\Delta}.$$

Since Δ, being the Laplace-Beltrami operator on the three-dimensional sphere, has a well-known heat kernel, it will lead to an expression of p_t.
Note also that in our cylindrical coordinates, the vector field Z has a particular simple expression since it reads

$$Z = \frac{\partial}{\partial z}.$$

The first think to do is then to find the expression of the heat kernel for the Laplace-Beltrami operator Δ on $SU(2)$ in our cylindrical coordinates. Let us consider on the interval $[-1, 1]$ the second order differential operator

$$\mathcal{J} = (1 - x^2)\frac{d^2}{dx^2} - 3x\frac{d}{dx}.$$

For $m \geq 0$, let U_m denotes the Chebyshev polynomial of the second kind:

$$U_m(\cos x) = \frac{\sin(m+1)x}{\sin x},$$

and

$$q_t(x) = \sum_{m=0}^{+\infty} (m+1)e^{-m(m+2)t} U_m(x), \quad x \in [-1, 1]. \tag{4.4.2}$$

It is known that if f is a smooth function $[-1, 1] \to \mathbb{R}$, then

$$(e^{t\mathcal{J}} f)(1) = \frac{2}{\pi}\int_{-1}^{1} q_t(x) f(x)(1 - x^2)^{1/2} dx,$$

Lemma 4.4.1. *If f is a smooth function $\mathbf{SU}(2) \to \mathbb{R}$, then for $t \geq 0$,*

$$(e^{t\Delta} f)(0) = \frac{1}{4\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{-\pi}^{\pi} q_t(\cos r \cos z) f(r, \theta, z) \sin 2r dr d\theta dz$$

Proof. An easy calculation shows that the function $q_t(\cos r \cos z)$ solves the heat equation

$$\frac{\partial}{\partial t}(q_t(\cos r \cos z)) = \Delta(q_t(\cos r \cos z)).$$

Now we have to check the initial condition. We must show

$$\frac{1}{\pi}\int_0^{\pi/2}\int_{-\pi}^{\pi} q_t(\cos r \cos z)\tilde{f}(r,z)\frac{\sin 2r}{2}drdz \longrightarrow_{t\to 0} \tilde{f}(0,0)$$

where $\tilde{f}(r,z) = \frac{1}{2\pi}\int f(r,\theta,z)d\theta$. Since we will make the following change of variables:

$$\begin{cases} u &= \cos r \cos z \\ v &= \cos r \sin z \end{cases}$$

we take the function $\tilde{f}$ of the form $\tilde{f}(r,z) = g(\cos r \cos z)h(\cos r \sin z)$. The new domain is $D = \{(x,y), x^2+y^2 \le 1\}$ and the Jacobian determinant is $\frac{1}{2}\sin 2r$. So

$$\begin{aligned} & \frac{1}{\pi}\int_{r=0}^{\pi/2}\int_{z=0}^{2\pi} q_t(\cos r \cos z)g(\cos r\cos z)h(\cos r \sin z)\frac{\sin 2r}{2}drdz \\ = & \frac{1}{\pi}\int\int_D q_t(u)g(u)h(v)dudv \\ = & \frac{1}{\pi}\int_{-1}^{1}\left(\int_{-(1-u^2)^{1/2}}^{(1-u^2)^{1/2}} h(v)dv\right) q_t(u)g(u)du \end{aligned}$$

We may rewrite it as

$$\frac{2}{\pi}\int_{-1}^{1} q_t(u)l(u)(1-u^2)^{1/2}du$$

where l is the continuous fonction

$$l(u) = g(u)\left(\frac{\int_{-(1-u^2)^{1/2}}^{(1-u^2)^{1/2}} h(v)dv}{2(1-u^2)^{1/2}}\right)$$

Now, since q_t is the heat kernel of a diffusion issued of 1 with respect to the measure $\frac{2}{\pi}(1-u^2)^{1/2}du$ and l is continuous, the last quantity is converging towards $l(1) = g(1)h(0) = \tilde{f}(0,0)$ and the lemma is proved. □

Remark 4.4.2. *The previous lemma shows that if ρ is the Riemannian distance from 0, then in our cylindrical coordinates, we have*

$$\cos\rho = \cos r \cos z.$$

From the previous proposition, we can now derive an expression for p_t in terms of q_t.
Let us first describe some properties of q_t that will be useful in the sequel. From the Poisson summation formula, we obtain that for $\theta \in \mathbb{R}$:

$$\begin{aligned} q_t(\cos\theta) &= \frac{\sqrt{\pi}e^t}{4t^{\frac{3}{2}}}\frac{1}{\sin\theta}\sum_{k\in\mathbb{Z}}(\theta+2k\pi)e^{-\frac{(\theta+2k\pi)^2}{4t}} \\ &= \frac{\sqrt{\pi}e^t}{4t^{\frac{3}{2}}}\frac{\theta}{\sin\theta}e^{-\frac{\theta^2}{4t}}\left(1+2\sum_{k=1}^{+\infty}e^{-\frac{k^2\pi^2}{t}}\left(\cosh\frac{k\pi\theta}{t}+2k\pi\frac{\sinh\frac{k\pi\theta}{t}}{\theta}\right)\right) \end{aligned}$$

These expressions show that $q_t(\cos\theta)$ admits an analytic extension for $\theta \in \mathbb{C} - (-\infty,-1]$. We moreover obtain precise estimates:

- Let $\varepsilon > 0$, for $x \in (-1+\varepsilon, 1]$ and $t > 0$:

$$q_t(x) = \frac{\sqrt{\pi}e^t}{4t^{\frac{3}{2}}} \frac{\text{arccos}x}{\sqrt{1-x^2}} e^{-\frac{(\text{arccos}x)^2}{4t}} \left(1 + R_1(t,x)\right), \tag{4.4.3}$$

  where for some positive constants C_1 and C_2 depending only in ε, $\mid R_1(t,x) \mid \leq C_1 e^{-\frac{C_2}{t}}$.

- For $x \in [1, +\infty)$ and $t > 0$:

$$q_t(x) = \frac{\sqrt{\pi}e^t}{4t^{\frac{3}{2}}} \frac{\text{arccosh}x}{\sqrt{x^2-1}} e^{\frac{(\text{arccosh}x)^2}{4t}} \left(1 + R_2(t,x)\right), \tag{4.4.4}$$

  where for some positive constants C_3 and C_4, $\mid R_2(t,x) \mid \leq C_3 e^{-\frac{C_4}{t}}$.

Proposition 4.4.3. *With respect to the measure $d\mu = \frac{\sin 2r}{2} dr d\theta dz$, the heat kernel associated to the semigroup $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ on $\mathbf{SU}(2)$ writes, for $t > 0$, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$,*

$$p_t(r,z) = \frac{1}{2\pi^2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+iz)^2}{4t}} q_t(\cos r \cosh y) dy$$

Proof. Let

$$r_t(r,z) = \frac{1}{2\pi^2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+iz)^2}{4t}} q_t(\cos r \cosh y) dy;$$

the integral being well defined thanks to the estimates on q_t. By using the fact that

$$\frac{\partial}{\partial t} \left(\frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi t}} \right) = \frac{\partial^2}{\partial y^2} \left(\frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi t}} \right)$$

and

$$\frac{\partial}{\partial t}(q_t(\cos r \cos z)) = \Delta(q_t(\cos r \cos z)),$$

a double integration by parts with respect to the variable y shows that

$$\frac{\partial r_t}{\partial t} = L r_t.$$

Let us now check the initial condition. Let $f(r,z) = e^{i\lambda z} g(r)$ where $\lambda \in \mathbb{R}$ and g is a smooth function. We have

$$\frac{1}{4\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\pi}^{\pi} r_t(r,z) f(r,z) \sin 2r dr d\theta dz = e^{t\lambda^2} \left(e^{t\Delta} g\right)(0),$$

so that we obtain the required result. $\square$

4.5 An integral representation for the heat kernel on $\mathbf{SL}(2,\mathbb{R})$

For the subelliptic heat kernel on $\mathbf{SL}(2,\mathbb{R})$, this time, from the pseudo-Riemannian submersion, we have a relation between the sublaplacian L and the Casimir operator $\Box$. Here we prefer to denote it by Δ_1. Δ_1 is in the center of the envelopping algebra of $\mathbf{SL}(2,\mathbb{R})$ (and if fact generates it). Δ_1 is not an elliptic operator but a hyperbolic operator. Recall the operators write:

$$L = X^2 + Y^2$$

and

$$\Delta_1 = X^2 + Y^2 - Z^2.$$

Recall also that

$$[L, Z] = 0.$$

Therefore one has the formula

$$e^{tL} = e^{tZ^2} e^{t\Delta_1}$$

from which one should be able to obtain a representation of the subelliptic heat kernel on $\mathbf{SL}(2,\mathbb{R})$. In fact, as we feel more comfortable with elliptic operators, we do not deduce this formula exactly in this way and we prefer to rely on heat kernel of the Laplace-Beltrami operator on the 3-dimensional hyperbolic space.

Let us consider the second order differential operator on the interval $[1,\infty)$

$$\mathcal{J} = (x^2-1)\frac{d^2}{dx^2} + 3x\frac{d}{dx}$$

with invariant and symmetric measure $(x^2-1)^{1/2}$. It is well known (see [89]) that the heat kernel s_t issued for 1 has the following expression for $x \geq 1$:

$$s_t(x) = \frac{e^{-t}}{\sqrt{4\pi}\, t^{3/2}} \left(\frac{\mathrm{arch} x}{\sqrt{x^2-1}}\right) e^{-\frac{(\mathrm{arch} x)^2}{4t}}. \tag{4.5.5}$$

That is, for f a smooth function $[1,\infty) \to \mathbb{R}$,

$$(e^{t\mathcal{J}} f)(1) = \int_1^{\infty} s_t(x) f(x)(x^2-1)^{1/2} dx.$$

It is clear the function $x \to (\mathrm{arch} x)^2$ admits an holomorphic extension to $\mathbb{C} - \{]\infty, 1]\}$; but in fact, using Schwarz symmetry principle (see [81] for instance), we can see that this extension is holmorphic on $\mathbb{C} - \{]\infty, -1]\}$. Therefore this is the same for its derivative: $x \to \frac{\mathrm{arch} x}{\sqrt{x^2-1}}$. So the heat kernel s_t itself admits an holomorphic extension to $\mathbb{C} - \{]\infty, -1]\}$. By setting $x = \cosh r, r \geq 0$, we have

$$s_t(\cosh r) = \frac{e^{-t}}{\sqrt{4\pi}\, t^{3/2}} \left(\frac{r}{\sinh r}\right) e^{-\frac{r^2}{4t}}. \tag{4.5.6}$$

This heat kernel corresponds in fact to the one on the 3-dimensionnal hyperbolic space.
Now easy calculations give us that s_t satisfies the following expressions:

$$\partial_t s_t(\cosh r \cos z) = \Delta_1 (s_t(\cosh r \cos z)) \tag{4.5.7}$$

where $\Delta_1 = \partial^2_{r,r} + 2\coth 2r\partial_r + (\tanh^2 r - 1)\partial^2_{z,z}$ and

$$\partial_t s_t(\cosh r \cosh y) = \Delta_2(s_t(\cosh r \cosh y)) \tag{4.5.8}$$

where $\Delta_2 = \partial^2_{r,r} + 2\coth 2r\partial_r + (1 - \tanh^2 r)\partial^2_{y,y}$.
Δ_1 and Δ_2 are two autoadjoint operators respectively on $(0,\infty)\times[-\pi,\pi]$ and on $(0,\infty)\times(0,\infty)$ with respective symmetric measure $\frac{\sinh 2r}{2}drdz$ and $\frac{\sinh 2r}{2}drdy$. Δ_1 is an hyperbolic operator whereas Δ_2 is an elliptic operator. As we said it before, Δ_1 is the Casimir operator on $\mathbf{SL}(2,\mathbb{R})$.

Now we will express the heat kernel for the operator Δ_2.

Lemma 4.5.1. *If f is a smooth function $(0,\infty)\times(0,2\pi)\times(0,\infty) \to \mathbb{R}$, then for $t \geq 0$,*

$$(e^{t\Delta_2} f)(0) = \frac{1}{2\pi}\int_{r>0}\int_{\theta=0}^{2\pi}\int_{y>0} s_t(\cosh r \cosh y) f(r,\theta,y)\frac{\sinh 2r}{2}drdy$$

Proof. Indeed we saw that s_t satisfies the equation:

$$\partial_t s_t(\cosh r \cosh y) = \Delta_2(s_t(\cosh r \cosh y)).$$

Now we must check the initial condition. As before we only have to show that for a smooth function f: $(0,\infty)\times(0,\infty) \to \mathbb{R}$:

$$\int_{r>0}\int_{y>0} s_t(\cosh r \cosh y) f(r,y)\frac{\sinh 2r}{2}drdy \to f(0,0) \text{ when } t \to 0.$$

Since we will make the following change of variables:

$$\begin{cases} u &= \cosh r \cosh y \\ v &= \cosh r \sinh y \end{cases}$$

we take the function f of the form $f(r,z) = g(\cosh r \cosh z)h(\cosh r \sinh z)$. The new domain is $D = \{(u,v), u \geq 1, v \geq 0, u^2 - v^2 \geq 1\}$ and the Jacobian determinant is $\frac{1}{2}\sinh 2r$. So

$$\begin{aligned} &\int_{r>0}\int_{y>0} s_t(\cosh r \cosh y)g(\cosh r \cosh y)h(\cosh r \sinh y)\frac{\sinh 2r}{2}drdy \\ = &\int\int_D s_t(u)g(u)h(v)dudv \\ = &\int_{u\geq 1}\left(\int_0^{(u^2-1)^{1/2}} h(v)dv\right) s_t(u)g(u)du \end{aligned}$$

We may rewrite it as

$$\int_{u\geq 1} s_t(u)l(u)(u^2-1)^{1/2}du$$

where l is the continuous fonction

$$l(u) = g(u)\left(\frac{\int_0^{(u^2-1)^{1/2}} h(v)dv}{(u^2-1)^{1/2}}\right)$$

Now, since s_t is the heat kernel of a diffusion issued of 1 with respect to the measure $(u^2-1)^{1/2}du$ and l is continuous, the last quantity is converging towards $l(1) = g(1)h(0) = f(0,0)$ and the lemma is proved. $\square$

Remark 4.5.2. *The function f in the lemma 4.5.1 is defined on the space $(0,\infty)\times(0,2\pi)\times(0,\infty)$ and not on $\mathbf{SL}(2,\mathbb{R})$ which is homemorphic to the space $(0,\infty)\times(0,2\pi)\times[-\pi,\pi]$.*

Proposition 4.5.3. *We have for $t>0, r>0, z\in[-\pi,\pi]$,*

$$\begin{aligned} p_t(r,z) &= \frac{1}{4\pi}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} s_t(\cosh r\cosh y)dy \\ &= \frac{e^{-t}}{(4\pi t)^2}\int_{-\infty}^{+\infty} e^{-\frac{\operatorname{arch}^2(\cosh r\cosh y)-(y-iz)^2}{4t}} \frac{\operatorname{arch}(\cosh r\cosh y)}{\sqrt{\cosh^2 r\cosh^2 y-1}}dy. \end{aligned}$$

Proof. The second equality is just obtain by using the explicit value of s_t and shows the integral is well defined since it is absolutely convergent. Now let

$$r_t(r,z)=\frac{1}{4\pi}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} s_t(\cosh r\cosh y)dy.$$

By using the fact that

$$\frac{\partial}{\partial t}\left(\frac{e^{\frac{(y-iz)^2}{4t}}}{\sqrt{4\pi t}}\right)=+\frac{\partial^2}{\partial z^2}\left(\frac{e^{\frac{(y-iz)^2}{4t}}}{\sqrt{4\pi t}}\right)=-\frac{\partial^2}{\partial y^2}\left(\frac{e^{\frac{(y-iz)^2}{4t}}}{\sqrt{4\pi t}}\right)$$

and

$$\frac{\partial}{\partial t}(s_t(\cosh r\cosh y))=\left(\partial^2_{r,r}+2\coth 2r\partial_r+(1-\tanh^2 r)\partial^2_{y,y}\right)(s_t(\cosh r\cosh y)),$$

a double integration by parts with respect to the variable y shows that

$$\frac{\partial r_t}{\partial t}=\frac{1}{4\pi}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}}\Delta_3(s_t(\cosh r\cosh y))dy$$

where $\Delta_3=\partial^2_{r,r}+2\coth 2r\partial_r-\tanh^2 r\partial^2_{y,y}$.
Now an other double integration by parts in the variable y shows us that

$$\frac{\partial}{\partial t}r_t(r,z)=Lr_t(r,z).$$

Let us now check the initial condition. Let $f(r,z)=e^{imz}g(r)$ where $m\in\mathbb{Z}$ and g is a smooth function. We have

$$\begin{aligned} &\int_{r>0}\int_{z=-\pi}^{\pi} r_t(r,z)f(r,z)\frac{\sinh 2r}{2}drdz \\ = &\frac{1}{4\pi}\int_{r>0}\int_{z=-\pi}^{\pi}\int_{y>0}\left(\frac{e^{-\frac{(z+iy)^2}{4t}}+e^{-\frac{(z-iy)^2}{4t}}}{\sqrt{4\pi t}}\right)s_t(\cosh r\cosh y)g(r)e^{imz}\frac{\sinh 2r}{2}drdzdy \end{aligned}$$

and by integrating in the z variable along a rectangle in the complex plane, we get

$$\begin{aligned} &\int_{z=-\pi}^{\pi}\left(\frac{e^{-\frac{(z+iy)^2}{4t}}}{\sqrt{4\pi t}}\right)e^{imz}dz \\ = \;& e^{my}\int_{z=-\pi}^{\pi}\left(\frac{e^{-\frac{z^2}{4t}}}{\sqrt{4\pi t}}\right)e^{imz}dz+2(-1)^m e^{-\frac{\pi^2}{4t}}\int_{u=0}^{y}\frac{e^{-\frac{(y-u)^2}{4t}}}{4\pi t}\sin\left(\frac{\pi(y-u)}{2t}\right)e^{mu}du \end{aligned}$$

We can do the same for the other term and eventually we obtain

$$\begin{aligned}
&\int_{r>0}\int_{z=-\pi}^{\pi} r_t(r,z)f(r,z)\frac{\sinh 2r}{2}drdz \\
&= \left(\int_{z=-\pi}^{\pi}\frac{e^{-\frac{z^2}{4t}}}{\sqrt{4\pi t}}e^{imz}dz\right)\int_{r>0}\int_{y>0}\frac{s_t(\cosh r\cosh y)}{2\pi}g(r)\frac{\cosh(my)}{2}\frac{\sinh 2r}{2}drdy \\
&+\frac{1}{2\pi}(-1)^m e^{-\frac{\pi^2}{4t}}\int_{r>0}\int_{y>0}\int_{u=0}^{y}\frac{e^{-\frac{(y-u)^2}{4t}}}{\sqrt{4\pi t}} \\
&\sin\left(\frac{\pi(y-u)}{2t}\right)\sinh(mu)s_t(\cosh r\cosh y)g(r)\frac{\sinh 2r}{2}dudrdy
\end{aligned}$$

The term in the first line is equal to $a(t)e^{t\Delta_2}(l)(0)$ where l is the function $l(r,y)=g(r)\frac{\cosh(my)}{2}$ and $a(t)=\int_{z=-\pi}^{\pi}\frac{e^{-\frac{z^2}{4t}}}{\sqrt{4\pi t}}e^{imz}dz$. By the classical heat kernel on $\mathbb{R}$, $a(t)$ tends to $e^{im0}=1$ when t goes to 0 and $e^{t\Delta_2}(l)(0)$ tends to $l(0)=g(0)$. Therefore, the first line term is converging to $g(0)=f(0,0)$ when t goes to 0. And one can check that the term on the second line is converging to 0 when t goes to 0. This gives us the desired convergence and ends our proof. $\square$

4.6 Some properties of the heat kernel

We are now in position to collect some properties of the heat kernel p_t on both spaces $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ by using the integral representation of the heat kernel.

4.6.1 The Laplace transform of p_t

The Heisenberg case

Proposition 4.6.1. *For $\lambda\in\mathbb{C}$, $\mathbf{Re}\lambda>0$, $r\geq 0$ and $z\in\mathbb{R}$,*

$$\int_0^{+\infty} h_t(r,z)e^{-\frac{\lambda}{t}}dt = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty}\frac{ydy}{\frac{r^2}{2}y\cosh y+(\lambda-iyz)\sinh y}$$

Proof. Indeed,

$$\begin{aligned}
\int_0^{\infty} h_t(r,z)e^{-\frac{\lambda}{t}}dt &= \frac{1}{16\pi^2}\int_{-\infty}^{+\infty}\frac{y}{\sinh y}\int_0^{\infty}\frac{1}{t^2}\exp\left(-\frac{1}{t}(\lambda-i\frac{yz}{2}+\frac{r^2}{4}y\coth y)\right)dtdy \\
&= \frac{1}{16\pi^2}\int_{-\infty}^{+\infty}\frac{y}{\sinh y}\frac{1}{\lambda-i\frac{yz}{2}+\frac{r^2}{4}y\coth y}dydt \\
&= \frac{1}{8\pi^2}\int_{-\infty}^{+\infty}\frac{ydy}{\frac{r^2}{2}y\cosh y+(\lambda-iyz)\sinh y}
\end{aligned}$$

$\square$

Inverting this Laplace transform gives another expression for the heat kernel.

Corollary 4.6.2. *Let* $r \geq 0$, $z \in \mathbb{R}$ *and* $\gamma \in \mathbb{C}$ *such* $Re(\gamma > 0)$, *then*

$$h_t(r,z) = \frac{1}{16 i \pi^3 t^2} \int_{\lambda=\gamma-i\infty}^{\gamma+i\infty} \int_{-\infty}^{+\infty} \frac{y e^{\frac{\lambda}{t}}}{\frac{r^2}{2} y \cosh y + (\lambda - iyz)\sinh y} d\lambda dy.$$

Note that if $(r,z) \neq (0,0)$, one can take also $\gamma = 0$ in the above corllary.
As another corollary of Proposition 4.6.1, we have:

Corollary 4.6.3. *The Green function of the operator* $-L$ *is given by*

$$G(r,z) = \frac{1}{8\pi} \frac{1}{\sqrt{\left(\frac{r^2}{2}\right)^2 + z^2}}.$$

for $(r,z) \neq (0,0)$.

Proof. Indeed, let $(r,z) \neq (0,0)$, then

$$\begin{aligned} G(r,z) &= (-L)^{-1} \\ &= \int_0^\infty e^{-t(-L)} dt \\ &= \int_0^\infty p_t(r,z) dt. \end{aligned}$$

Since $(r,z) \neq (0,0)$, the computation of Proposition 4.6.1 is still valid with $\lambda = 0$ and gives

$$\begin{aligned} G(r,z) &= \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} \frac{y dy}{\frac{r^2}{2} y \cosh y - iyz \sinh y} \\ &= \frac{1}{8\pi^2} \frac{1}{\sqrt{\left(\frac{r^2}{2}\right)^2 + z^2}} \int_{-\infty}^{+\infty} \frac{dy}{\cosh(y + \alpha(r,z))} \\ &= \frac{1}{8\pi^2} \frac{1}{\sqrt{\left(\frac{r^2}{2}\right)^2 + z^2}} \int_{-\infty}^{+\infty} \frac{dy}{\cosh(y)} \\ &= \frac{1}{8\pi} \frac{1}{\sqrt{\left(\frac{r^2}{2}\right)^2 + z^2}}. \end{aligned}$$

□

The SU(2) case

Proposition 4.6.4. *For* $\lambda \in \mathbb{C}$, $\mathbf{Re}\lambda > 0$, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$,

$$\int_0^{+\infty} p_t(r,z) e^{-t-\frac{\lambda}{t}} dt = \int_{-\infty}^{+\infty} \frac{dy}{8\pi^2 \left(\cosh \sqrt{y^2 + 4\lambda} - \cos r \cos(z + iy) \right)}$$

Proof. We have

$$\int_0^{+\infty} e^{-\frac{\lambda}{t}} p_t(r,z) e^{-t} dt = \frac{1}{2\pi^2}\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-t-\frac{y^2+4\lambda}{4t}} q_t(\cos r \cos(z+iy)) \frac{dt}{\sqrt{t}} dy$$

We now compute

$$\frac{1}{2\pi^2} \int_0^{+\infty} e^{-t-\frac{y^2+4\lambda}{4t}} q_t(\cos r \cos(z+iy)) \frac{dt}{\sqrt{t}}$$

by using the symbolic calculus on differential operators (it can be made rigorous with 4.4.2) and the subordination identity (see [88] identity 5.3.1)

$$\frac{e^{-\lambda x}}{\lambda} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{x^2}{4t}} e^{-\lambda^2 t} \frac{dt}{\sqrt{t}} \text{ for } \lambda, x > 0 \tag{4.6.9}$$

to obtain

$$\begin{aligned}\int_0^{+\infty} e^{-t-\frac{y^2+4\lambda}{4t}} e^{t\Delta_{S^3}} \frac{dt}{\sqrt{t}} &= \int_0^{+\infty} e^{-\frac{y^2+4\lambda}{4t}} e^{-t(-\Delta_{S^3}+1)} \frac{dt}{\sqrt{t}} \\ &= \frac{\sqrt{\pi}}{\sqrt{-\Delta_{S^3}+1}} e^{-\sqrt{y^2+4\lambda}\sqrt{-\Delta_{S^3}+1}}\end{aligned}$$

But from Taylor [89] pp. 95,

$$\frac{1}{\sqrt{-\Delta_{S^3}+1}} e^{-\sqrt{y^2+4\lambda}\sqrt{-\Delta_{S^3}+1}} = \frac{1}{4\pi^2 \left(\cosh \sqrt{y^2+4\lambda} - \cos r \cos(z+iy)\right)},$$

which implies the result. □

If we fix, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$, we observe that it is possible to find $\theta(r,z) \in \mathbb{R}$, such that for $\lambda \in \mathbb{C}, \mathbf{Re}\lambda > 0$ and $y \in \mathbb{R}$,

$$\cosh \sqrt{y^2+4\lambda} = \cos r \cos(z+iy) \Rightarrow \mathbf{Re}\lambda \le \theta(r,z),$$

where we use the principal branch of the square root. By inverting the last Laplace transform of the previous proposition, we therefore get:

Corollary 4.6.5. *We have for $t > 0$, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$, and $\gamma > \theta(r,z)$,*

$$p_t(r,z) = \frac{e^t}{16 i \pi^3 t^2} \int_{\lambda=\gamma-i\infty}^{\gamma+i\infty} \int_{y=-\infty}^{+\infty} \frac{e^{\frac{\lambda}{t}} dy d\lambda}{\cosh \sqrt{y^2+4\lambda} - \cos r \cos(z+iy)}$$

From Proposition 4.6.4, we also deduce:

Proposition 4.6.6. *The Green function of the operator $-L+1$ is given by*

$$G(r,z) = \frac{1}{8\pi} \frac{1}{\sqrt{1 - 2\cos r \cos z + \cos^2 r}}.$$

Proof. Let us assume $r \neq 0$, $z \neq 0$. In that case the Laplace transform of Proposition 4.6.4 can be extended to $\lambda = 0$ and we have:

$$\begin{aligned}
G(r,z) &= (-L+1)^{-1} \\
&= \int_0^{\infty} e^{-t(-L+1)} dt \\
&= \int_0^{\infty} p_t(r,z) e^{-t} dt \\
&= \int_{-\infty}^{+\infty} \frac{dy}{8\pi^2 \left(\cosh y - \cos r \cos(z+iy)\right)} \\
&= \int_{-\infty}^{+\infty} \frac{dy}{8\pi^2 \left((1-\cos r \cos z)\cosh y + i \cos r \sin z \sinh y\right)} \\
&= \frac{1}{8\pi^2} \frac{1}{\sqrt{1 - 2\cos r \cos z + \cos^2 r}} \int_{-\infty}^{+\infty} \frac{dy}{\cosh y} \\
&= \frac{1}{8\pi} \frac{1}{\sqrt{1 - 2\cos r \cos z + \cos^2 r}}.
\end{aligned}$$

□

The SL$(2,\mathbb{R})$ case

Proposition 4.6.7. *For $\lambda \in \mathbb{C}$, $\mathbf{Re}\lambda > 0$, $r \geq 0$, $z \in [-\pi, \pi]$,*

$$\int_0^{+\infty} p_t(r,z) e^{t+\frac{\lambda}{t}} dt = \int_{-\infty}^{+\infty} \frac{dy}{8\pi^2 \left(\cosh \sqrt{y^2+4\lambda} - \cosh r \cosh(y+iz)\right)}$$

Proof. We have

$$\int_0^{+\infty} e^{\frac{\lambda}{t}} p_t(r,z) e^t dt = \frac{1}{(\sqrt{4\pi})^{3/2}} \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{t+\frac{y^2+4\lambda}{4t}} s_t(\cos r \cos(z+iy)) \frac{dt}{\sqrt{t}} dy$$

We now compute

$$\frac{1}{4\pi} \int_0^{+\infty} e^{t\frac{y^2+4\lambda}{4t}} s_t(\cos r \cos(z+iy)) \frac{dt}{\sqrt{t}}$$

by using the symbolic calculus on differential operators

$$\begin{aligned}
\int_0^{+\infty} e^{t+\frac{y^2+4\lambda}{4t}} e^{t\Delta_{H^3}} \frac{dt}{\sqrt{t}} &= \int_0^{+\infty} e^{\frac{y^2+4\lambda}{4t}} e^{-t(-\Delta_{H^3}-1)} \frac{dt}{\sqrt{t}} \\
&= \frac{\sqrt{\pi}}{\sqrt{-\Delta_{H^3}-1}} e^{\sqrt{y^2+4\lambda}\sqrt{-\Delta_{H^3}+1}}
\end{aligned}$$

But from Taylor [89],

$$\frac{1}{\sqrt{-\Delta_{H^3}-1}} e^{\sqrt{y^2+4\lambda}\sqrt{-\Delta_{H^3}-1}} = \frac{1}{4\pi^2 \left(\cosh \sqrt{y^2+4\lambda} - \cosh r \cosh(y+iz)\right)},$$

which implies the result. □

If we fix, $r > 0$, $z \in [-\pi, \pi]$, we observe that it possible to find $\theta(r, z) \in \mathbb{R}$, such that for $\lambda \in \mathbb{C}, \mathbf{Re}\lambda > 0$ and $y \in \mathbb{R}$,

$$\cosh \sqrt{y^2 + 4\lambda} = \cosh r \cosh(y + iz) \Rightarrow \mathbf{Re}\lambda \leq \theta(r, z),$$

where we use the principal branch of the square root. By inverting the last Laplace transform of the previous proposition, we therefore get:

Corollary 4.6.8. *We have for $t > 0$, $r \in [0, \pi/2)$, $z \in [-\pi, \pi]$, and $\gamma > \theta(r, z)$,*

$$p_t(r, z) = \frac{e^t}{16 i \pi^3 t^2} \int_{\lambda=\gamma-i\infty}^{\gamma+i\infty} \int_{y=-\infty}^{+\infty} \frac{e^{\frac{\lambda}{t}} dy d\lambda}{\cosh \sqrt{y^2 + 4\lambda} - \cosh r \cosh(y + iz)}$$

From Proposition 4.6.7, we also deduce:

Proposition 4.6.9. *The Green function of the operator $-L - 1$ is given by*

$$G(r, z) = \frac{1}{8\pi} \frac{1}{\sqrt{1 - 2\cosh r \cos z + \cosh^2 r}}.$$

Proof. Let us assume $r \neq 0$, $z \neq 0$. In that case the Laplace transform of Proposition 4.6.7 can be extended to $\lambda = 0$ and we have:

$$\begin{aligned}
G(r, z) &= \int_0^{\infty} e^{-t(-L-1)} dt \\
&= \int_0^{\infty} p_t(r, z) e^t dt \\
&= \int_{-\infty}^{+\infty} \frac{dy}{8\pi^2 (\cosh y - \cosh r \cosh(y + iz))} \\
&= \int_{-\infty}^{+\infty} \frac{dy}{8\pi^2 ((1 - \cosh r \cos z)\cosh y - i \cosh r \sin z \sinh y)} \\
&= \frac{1}{8\pi^2} \frac{1}{\sqrt{1 - 2\cosh r \cos z + \cosh^2 r}} \int_{-\infty}^{+\infty} \frac{dy}{\cosh y} \\
&= \frac{1}{8\pi} \frac{1}{\sqrt{1 - 2\cosh r \cos z + \cosh^2 r}}
\end{aligned}$$

□

4.6.2 Asymptotics of the heat kernel in small times

The goal of this section is to obtain the precise asymptotics of the heat kernel when $t \to 0$ on the three spaces $\mathbb{H}$, $\mathbf{SU}(2)$ and $\mathbf{SL}(2, \mathbb{R})$.

The Heisenberg case

We start with the points of the form $(0, z)$ that lie on the cut-locus of 0. For simplicity we restrict ourself to points such that $z > 0$. The heat kernel at this point $(0, z)$ reads

$$h_t(0, z) = \frac{1}{16\pi^2 t^2} \int_{-\infty}^{+\infty} e^{\frac{i\lambda z}{2t}} \frac{\lambda}{\sinh \lambda} d\lambda.$$

A computation of this integral is possible using residus calculus and one gets

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{\frac{i\lambda z}{2t}} \frac{\lambda}{\sinh\lambda} d\lambda &= 2i\pi \sum_{k\geq 1} (-1)^k (ik\pi) e^{-k\frac{z\pi}{2t}} \\ &= 2\pi^2 \frac{e^{-\frac{\pi z}{2t}}}{\left(1+e^{-\frac{\pi z}{2t}}\right)^2}.\end{aligned}$$

Therefore:

Proposition 4.6.10. *Let $z > 0$,*

$$h_t(0,z) = \frac{1}{8t^2} \frac{e^{-\frac{\pi z}{2t}}}{\left(1+e^{-\frac{\pi z}{2t}}\right)^2}. \tag{4.6.10}$$

By continuity of the heat kernel, one can obtain the value in 0.

Proposition 4.6.11.

$$h_t(0,0) = \frac{1}{32t^2}. \tag{4.6.11}$$

We now start to study the points which are not on the cut locus. We begin by studying the points $(r,0)$ with $r > 0$. For these points the heat kernel reads:

$$h_t(r,0) = \frac{1}{16\pi^2 t^2} \int_{-\infty}^{+\infty} e^{-\frac{r^2}{4t} y \coth y} \frac{y}{\sinh y} dy.$$

The function $f(y) = r^2 y \coth y$ has a unique minimum at the point $y = 0$ and at this point:

$$f''(0) = \frac{2r^2}{3}.$$

But, from the Laplace method, for g a smooth function such that $g(0) \neq 0$, when t goes to 0:

$$\int_{\mathbb{R}} e^{-\frac{f(y)}{4t}} g(y) dy \sim \sqrt{\frac{2\pi}{f''(0)}} \sqrt{t} e^{-\frac{f(0)}{4t}} g(0). \tag{4.6.12}$$

From which, the next proposition follows:

Proposition 4.6.12. *For $r > 0$, when t goes to 0,*

$$h_t(r,0) \sim \frac{1}{2} \frac{1}{(4\pi t)^{3/2}} \frac{\sqrt{3}}{r} e^{-\frac{r^2}{4t}}.$$

Now, by the same method, we can extend the result to the points (r,z), $r > 0$ and $z \neq 0$. Here we assume $z > 0$. The function

$$f : y \to -iy\frac{z}{2} + \frac{r^2}{4} y \coth y$$

is mermorphic on $\mathbb{C}$ with poles in $ik\pi$, $k \in \mathbb{Z} - \{0\}$. In the strip $|Im(y)| < \pi$, it has an unique critical point. This critical point is $i\theta(r,z)$ where θ is the unique solution in $(0,\pi)$ of the equation

$$\left(\frac{\theta(r,z)}{\sin^2\theta(r,z)} - \operatorname{cotan}\theta(r,z)\right) r^2 = 2z.$$

The second derivative at this point is a positive real number:

$$f''(i\theta(r,z)) = \frac{r^2}{2}\left(\frac{\theta(r,z)\cos\theta(r,z)-\sin\theta(r,z)}{\sin^3\theta(r,z)}\right).$$

This and the Laplace method give the following proposition:

Proposition 4.6.13. *For $r>0$ and $z>0$, when t goes to 0:*

$$h_t(r,z) \sim \frac{1}{(4\pi t)^{3/2}}\frac{\sin\theta(r,z)}{r}\sqrt{\frac{\sin\theta(r,z)}{\theta(r,z)\cos\theta(r,z)-\sin\theta(r,z)}}\,e^{-\left(\frac{r^2\theta(r,z)\cot\theta(r,z)-2z\theta(r,z)}{4t}\right)}.$$

Heat kernel estimates on the Heisenberg group

In fact, by developping this method further and making a more carefull analysis of the meromorphic function of the integral, it is possible to obtain optimal estimates for the heat kernel (see [19, 70, 56], see also Sections 7.3.2 and 7.3.3):

Proposition 4.6.14. *There exists a constant $C>0$ such that for all $r>0, z\in\mathbb{R}$,*

$$\frac{1}{C}\frac{\exp\frac{-d^2(r,z)}{4t}}{\sqrt{t^4+t^3rd(r,z)}} \le p_t(r,z) \le C\frac{\exp\frac{-d^2(r,z)}{4t}}{\sqrt{t^4+t^3rd(r,z)}}. \tag{4.6.13}$$

One can also obtain estimates of some gradients of the heat kernel (see [70]):

Proposition 4.6.15.

$$\Gamma(\log h_t)(r,z) \le C\frac{d^2(r,z)}{t^2} \tag{4.6.14}$$

and

$$|Z(\log h_t)(r,z)|^2 \le \frac{C}{t^2}. \tag{4.6.15}$$

The last one 4.6.15 is not completely explicited in [70] but follows easily from the estimation of W_1 page 376 of this paper.

The SU(2) case

As before, we start with the points of the form $(0,z)$ that lie on the cut-locus of 0.

Proposition 4.6.16. *On $\mathbf{SU}(2)$, for $t>0$ and $z\in(0,\pi)$,*

$$p_t(0,z) = \frac{e^t}{8t^2}e^{-\frac{2\pi z-z^2}{4t}}\sum_{k\in\mathbb{Z}}e^{-\frac{k(k+1)\pi^2}{t}}\frac{(2k+1)+2ke^{-\frac{\pi}{2t}(z+2k\pi)}}{\left(1+e^{-\frac{\pi}{2t}(z+2k\pi)}\right)^2}$$

therefore, when $t\to 0$,

$$p_t(0,z) = \frac{e^t}{8t^2}\frac{e^{-\frac{2\pi z-z^2}{4t}}}{\left(1+e^{-\frac{\pi z}{2t}}\right)^2}\left(1+O(e^{-\frac{C}{t}})\right)$$

Proof. Let $z \in (0, \pi]$. We have

$$p_t(0,z) = \frac{1}{2\pi^2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4t}} q_t(\cosh(y - iz)))dy,$$

But

$$q_t(\cosh(y-iz)) = \frac{\sqrt{\pi} e^t}{4t^{\frac{3}{2}}} \frac{1}{\sinh(y-iz)} \sum_{k \in \mathbb{Z}} (y - iz - 2ik\pi) e^{\frac{(y-iz-2ik\pi)^2}{4t}}$$

and for $k \in \mathbb{Z}$, from the residue theorem,

$$\int_{-\infty}^{+\infty} \frac{y - iz - 2ik\pi}{\sinh(y-iz)} e^{-\frac{iy}{2t}(z+2k\pi)} dy = 2\pi^2 e^{\frac{(z+2k\pi)^2 - (2k+1)\pi(z+2k\pi)}{2t}} \frac{(2k+1) + 2ke^{-\frac{\pi}{2t}(z+2k\pi)}}{\left(1 + e^{-\frac{\pi}{2t}(z+2k\pi)}\right)^2}.$$

The result easily follows. □

We now come to points (r, z) that do not lie on the cut-locus, that is $r \neq 0$.

Proposition 4.6.17. *On* $\mathbf{SU}(2)$, *for* $r \in \left(0, \frac{\pi}{2}\right)$, *when* $t \to 0$,

$$p_t(r, 0) \sim \frac{r}{\sin r} \sqrt{\frac{1}{1 - r\text{cotan}r}} \frac{e^{-\frac{r^2}{4t}}}{2(4\pi t)^{\frac{3}{2}}}.$$

Proof. We fix $r \in \left(0, \frac{\pi}{2}\right)$. From the proposition 4.4.3 and due to the estimates on q_t we get:

$$p_t(r, 0) \sim_{t \to 0} \frac{1}{16\pi^2 t^2} (J_1(t) + J_2(t)),$$

where

$$J_1(t) = \int_{\cosh y \leq \frac{1}{\cos r}} e^{-\frac{y^2 + (\arccos(\cos r \cosh y))^2}{4t}} \frac{\arccos(\cos r \cosh y)}{\sqrt{1 - \cos^2 r \cosh^2 y}} dy$$

and

$$J_2(t) = \int_{\cosh y \geq \frac{1}{\cos r}} e^{-\frac{y^2 - (\text{arcosh}(\cos r \cosh y))^2}{4t}} \frac{\text{arcosh}(\cos r \cosh y)}{\sqrt{\cos^2 r \cosh^2 y - 1}} dy.$$

We now analyze the two above integrals in small times thanks to the Laplace method and show that $J_2(t)$ can be omitted.
On the interval $\left[-\text{arcosh}\frac{1}{\cos r}, \text{arcosh}\frac{1}{\cos r}\right]$, the function

$$f(y) = y^2 + (\arccos(\cos r \cosh y))^2$$

has a unique minimum which is attained at $y = 0$ and, at this point:

$$f''(0) = 2(1 - r\text{cotan}r).$$

Therefore, thanks to the Laplace method

$$J_1(t) \sim_{t \to 0} e^{-\frac{r^2}{4t}} \frac{r}{\sin r} \sqrt{\frac{\pi t}{1 - r\text{cotan}r}}.$$

We now analyze the second integral. On $\left(-\infty, -\text{arcosh}\frac{1}{\cos r}\right) \cup \left(\text{arcosh}\frac{1}{\cos r}, +\infty\right)$, the function

$$g(y) = y^2 - (\text{arcosh}(\cos r \cosh y))^2,$$

has no minimum. Therefore, from the Laplace method $J_2(t)$ is negligeable with respect to $J_1(t)$ when $t \to 0$. □

The previous proposition can be extended by the same method when $z \neq 0$. If we fix $r \in \left(0, \frac{\pi}{2}\right), z \in [-\pi, \pi]$, then the function

$$f(y) = (y - iz)^2 + (\text{arcos}(\cos r \cosh y))^2,$$

defined on the strip $\mid \text{Re}(y) \mid < \text{arcosh}\frac{1}{\cos r}$ has a critical point at $i\theta(r, z)$ where $\theta(r, z)$ is the unique solution in $[-\pi, \pi]$ to the equation:

$$\theta(r, z) - z = \cos r \sin \theta(r, z) \frac{\text{arcos}(\cos \theta(r, z) \cos r)}{\sqrt{1 - \cos^2 r \cos^2 \theta(r, z)}}.$$

Indeed, with $u = \cos r \cos \theta$

$$\frac{\partial}{\partial \theta}\left(\theta - \cos r \sin \theta \frac{\text{arcos}(\cos \theta \cos r)}{\sqrt{1 - \cos^2 r \cos^2 \theta}}\right) = \frac{\sin^2 r}{1 - u(r, z)^2}\left(1 - \frac{u(r, z)\text{arcos}u(r, z)}{\sqrt{1 - u^2(r, z)}}\right)$$

which is positive and actually bigger than 1. So this last function is bijective from $[-\pi, \pi]$ on itself.
We observe that at the point $\theta(r, z)$, $f''(i\theta(r, z))$ is a positive real number:

$$f''(i\theta(r, z)) = 2\frac{\sin^2 r}{1 - u(r, z)^2}\left(1 - \frac{u(r, z)\text{arcos}u(r, z)}{\sqrt{1 - u^2(r, z)}}\right)$$

where $u(r, z) = \cos r \cos \theta(r, z)$. By the same method than in the previous proposition, we obtain:

Proposition 4.6.18. *On* $\mathbf{SU}(2)$, *let* $r \in \left(0, \frac{\pi}{2}\right), z \in [-\pi, \pi]$. *When* $t \to 0$,

$$p_t(r, z) \sim \frac{1}{\sin r} \frac{\text{arccos}u(r, z)}{\sqrt{1 - \frac{u(r,z)\text{arcos}u(r,z)}{\sqrt{1-u^2(r,z)}}}} \frac{e^{-\frac{(\theta(r,z)-z)^2 \tan^2 r}{4t \sin^2 \theta(r,z)}}}{(4\pi t)^{\frac{3}{2}}}.$$

The $\mathbf{SL}(2, \mathbb{R})$ case

We can now do the same on $\mathbf{SL}(2, \mathbb{R})$. As before, we start with the points of the form $(0, z)$ that lie on the cut-locus of 0. We restrict ourselves to the points with $z > 0$. For these points we have

$$p_t(0, z) = \frac{e^{-t}}{(4\pi t)^2} e^{-\frac{z^2}{4t}} \int_{-\infty}^{+\infty} e^{\frac{-iyz}{2t}} \frac{y}{\sinh y} dy.$$

A computation of the integral is possible using residus calculus and gives the following:

Proposition 4.6.19. *For* $z \in (0, \pi]$ *and* $t > 0$,

$$p_t(0, z) = \frac{e^{-t}}{8t^2} \frac{e^{-\frac{2\pi z + z^2}{4t}}}{\left(1 + e^{-\frac{\pi z}{2t}}\right)^2}$$

therefore, there exists a constant C *such that when* $t \to 0$, *for* $z \in (0, \pi]$

$$p_t(0, z) = \frac{e^{-t}}{8t^2} e^{-\frac{2\pi z + z^2}{4t}} \left(1 + O(e^{-\frac{C}{t}})\right)$$

By continuity of the heat kernel we obtain the value on the diagonal.

Proposition 4.6.20. *For $t > 0$,*

$$p_t(0,0) = \frac{e^{-t}}{32t^2}.$$

Now we turn to points of the form $(r,0)$ and give their asymptotics for the heat kernel.

Proposition 4.6.21. *For $r > 0$, when $t \to 0$,*

$$p_t(r,0) \sim \frac{1}{2(4\pi t)^{\frac{3}{2}}} \frac{r}{\sinh r} \sqrt{\frac{1}{r \coth r - 1}} e^{-\frac{r^2}{4t}}.$$

Proof. We have for $r > 0$

$$p_t(r,0) = \frac{e^{-t}}{(4\pi t)^2} \int_{-\infty}^{+\infty} e^{-\frac{\operatorname{arch}^2(\cosh r \cosh y) - y^2}{4t}} \frac{\operatorname{arch}(\cosh r \cosh y)}{\sqrt{\cosh^2 r \cosh^2 y - 1}} dy$$

We now analyze the above integral in small times thanks to the Laplace method.
On $\mathbb{R}$, the function

$$f(y) = \operatorname{arch}(\cosh r \cosh y))^2 - y^2$$

has a unique minimum which is attained at $y = 0$ and is equal to r^2, at this point:

$$f''(0) = 2(r \coth r - 1).$$

The result follows by the Laplace method. $\square$

The previous proposition can be extended by the same method when $z \neq 0$. Let $r > 0, z \in [-\pi, \pi]$ and consider the function

$$f(y) = (\operatorname{arch}(\cosh r \cosh y))^2 - (y - iz)^2,$$

This function is well defined and holomorphic on the strip $|\mathrm{Im}(y)| < \arccos\left(\frac{-1}{\cosh r}\right)$ and it has a critical point at $i\theta(r,z)$ where $\theta(r,z)$ is the unique solution in $\left(-\arccos\left(\frac{-1}{\cosh r}\right), \arccos\left(\frac{-1}{\cosh r}\right)\right)$ to the equation:

$$\theta(r,z) - z = \cosh r \sin\theta(r,z) \frac{\operatorname{arch}(\cosh r \cos\theta(r,z))}{\sqrt{\cosh^2 r \cos^2\theta(r,z) - 1}}.$$

Indeed the function $\theta \to \cosh r \sin\theta(r,z) \frac{\operatorname{arch}(\cosh r \cos\theta(r,z))}{\sqrt{\cosh^2 r \cos^2\theta(r,z) - 1}}$ is continuous, strictly increasing from $-\infty$ to ∞ and with a derivative greater than 1.
At the critical point, $f''(i\theta(r,z))$ is a positive and real number

$$f''(i\theta(r,z)) = 2\frac{\sinh^2 r}{u(r,z)^2 - 1}\left[\frac{u(r,z)\operatorname{arch}u(r,z)}{\sqrt{u(r,z)^2 - 1}} - 1\right]$$

with $u(r,z) = \cosh r \cos\theta(r,z)$ since $u > -1$.
We may observe that z and $\theta(r,z)$ have opposite signs.
By the same method than in the previous proposition, we obtain:

Proposition 4.6.22. *Let $r > 0, z \in [-\pi, \pi]$. When $t \to 0$,*

$$p_t(r,z) \sim \frac{1}{\sinh r} \frac{\operatorname{arccosh} u(r,z)}{\sqrt{\frac{u(r,z)\operatorname{arcosh} u(r,z)}{\sqrt{u^2(r,z) - 1}} - 1}} \frac{e^{-\frac{(\theta(r,z) - z)^2 \tanh^2 r}{4t \sin^2\theta(r,z)}}}{(4\pi t)^{\frac{3}{2}}}$$

with $u(r,z) = \cosh r \cos\theta(r,z)$.

4.6.3 The computation of the subriemannian distance

According to Léandre results [65] and [64] (see also [51]), the previous asymptotics give a way to compute the sub-Riemannian distance from 0 to the point $(r,\theta,z) \in \mathbb{G}$ by computing $\lim_{t\to 0} -4t \ln p_t(r,z)$. This distance does not depend on the variable θ and shall be denoted by $d(r,z)$.

The Heisenberg case

Proposition 4.6.23. *On $\mathbb{H}$, one has*

- *For $z \in \mathbb{R}$,*
$$d^2_{\mathbb{H}}(0,z) = 2\pi \mid z \mid .$$
- *For $r > 0$,*
$$d^2_{\mathbb{H}}(r,0) = r^2.$$
- *For $r > 0$, $z \in \mathbb{R}$*
$$d^2_{\mathbb{H}}(r,z) = r^2\theta(r,z)\cot\theta(r,z) - 2z\theta(r,z).$$

From this proposition, we can get some estimates of the distance:

Proposition 4.6.24. *There exists two constants $c, C > 0$ such that for all $r > 0$ and $z \in [-\pi,\pi]$:*
$$c(r^2+|z|) \le d^2(r,z) \le C(r^2+|z|).$$

For the Heisenberg group, this fact is well known and can also be easily obtained by the use of the dilations dil_λ since $\sqrt{r^2+|z|}$ is an homogenous norm. But as we said, this fact can also be obtained by the previous estimates. Here we do not do this proof but it is very similar to the proofs of the same proposition on $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$.

The $\mathbf{SU}(2)$ case

Proposition 4.6.25. *On $\mathbf{SU}(2)$, one has*

- *For $z \in [-\pi,\pi]$,*
$$d^2_{\mathbf{SU}(2)}(0,z) = 2\pi \mid z \mid -z^2.$$
- *For $r > 0$,*
$$d^2_{\mathbf{SU}(2)}(r,0) = r^2.$$
- *For $z \in [-\pi,\pi]$, $r \in \left(0,\frac{\pi}{2}\right)$,*
$$d^2_{\mathbf{SU}(2)}(r,z) = \frac{(\theta(r,z)-z)^2\tan^2 r}{\sin^2\theta(r,z)}.$$

Remark 4.6.26. *We can observe that the subriemannian diameter of $\mathbf{SU}(2)$ is thus π.*

From this proposition, we can get the estimates of the distance:

Proposition 4.6.27. *There exists two constants $c, C > 0$ such that for all $r > 0$ and $z \in [-\pi, \pi]$:*

$$c(r^2 + |z|) \leq d^2(r, z) \leq C(r^2 + |z|).$$

Proof. For the right inequality, as in our coordinates on the group $\mathbf{SU}(2)$, $(r, 0, 0) * (0, 0, z) = (r, 0, z)$, we obtain by using the left invariance of the distance: $d(r, z) \leq d(r, 0) + d(0, z)$. By combining it with the previous result, for all $r > 0$ and $z \in [-\pi, \pi]$, we get:

$$d^2(r, z) \leq C(r^2 + |z|)$$

where C is a positive constant.
Let us turn to the left inequality. Since $(r, 0, z)*(0, 0, -z) = (r, 0, 0)$, then $d(r, 0)-d(0, z) \leq d(r, z)$ and so the result is true in the region where $r^2 \geq A|z|$ with A big enough.
Similarly, since $(r, \pi, 0) * (r, 0, z) = (0, 0, z)$ then $d(0, z) - d(r, 0) \leq d(r, z)$ and the result is true in the region where $|z| \geq Br^2$ with B big enough. Now, consider the region $\{(r, z), \frac{1}{A}r^2 \leq |z| \leq Br^2\}$. Since $|z|$ is bounded by π, r is also bounded above on this region.
Recall now, using the equation of the critical point, that the distance is given by

$$\begin{aligned} d^2(r, z) &= \sin^2 r \left(\frac{\arccos(\cos\theta(r, z) \cos r)}{\sqrt{1 - \cos^2 r \cos^2 \theta(r, z)}} \right)^2 \\ &\geq r^2 \text{ for } 0 < r < \frac{\pi}{2}. \end{aligned}$$

In the region consider, r^2 and $|z|$ are of the same order which ends the proof. □

The $\mathbf{SL}(2, \mathbb{R})$ case

Proposition 4.6.28. *On $\mathbf{SL}(2, \mathbb{R})$, one has*

- *For $z \in [-\pi, \pi]$,*
$$d^2_{\mathbf{SL}(2,\mathbb{R})}(0, z) = 2\pi \mid z \mid +z^2.$$
- *For $r > 0$,*
$$d^2_{\mathbf{SL}(2,\mathbb{R})}(r, 0) = r^2.$$
- *For $z \in [-\pi, \pi]$, $r > 0$,*
$$d^2_{\mathbf{SL}(2,\mathbb{R})}(r, z) = \frac{(\theta(r, z) - z)^2 \tanh^2 r}{\sin^2 \theta(r, z)}.$$

From this proposition, we can also get the same estimates of the distance:

Proposition 4.6.29. *There exists two constants $c, C > 0$ such that for all $r > 0$ and $z \in [-\pi, \pi]$:*

$$c(r^2 + |z|) \leq d(r, z) \leq C(r^2 + |z|).$$

Proof. The right inequality follows the same line as the one on $\mathbf{SU}(2)$ since in our coordinates on the group $\mathbf{SL}(2, \mathbb{R})$, the same formula for the product holds: $(r, 0, 0) * (0, 0, z) = (r, 0, z)$
For the left inequality, we can also restrict ourselt to the region $\{(r, z), \frac{1}{A}r^2 \leq |z| \leq Br^2\}$ since the product formulas $(r, 0, z) * (0, 0, -z) = (r, 0, 0)$ and $(r, \pi, 0) * (r, 0, z) = (0, 0, z)$ still hold on $\mathbf{SL}(2, \mathbb{R})$. Note that since z is bounded, r is also bounded above in this region.

Now, reminding that the critical point $\theta(r,z)$ and z have opposite signs, one has

$$d^2(r,z) = \frac{(\theta(r,z)-z)^2}{\sin^2\theta(r,z)}\tanh^2 r \geq (1+2|z|)\tanh^2 r.$$

But as r is bounded above, there exists a constant c' such that $\tanh^2 r \geq c' r^2$. So on this domain:

$$d^2(r,z) \geq c' r^2(1+|z|).$$

On this domain the function on the right side behaves like $r^2+|z|$ and gives the result. □

An ultracontrative bound on $\mathbf{SL}(2,\mathbb{R})$

The proposition 4.6.20 gives that the heat kernel satisfies the following ultracontractivity bound:

$$p_t(0,0) = ||p_t||_\infty \leq \frac{e^{-t}}{32t^2}. \tag{4.6.16}$$

Now by using well known results from Davies (see [36] or [92]), this leads to the following general gaussian upper estimate (where we do not take into account the exponential decay):

$$p_t(r,z) \leq \frac{C_\eta}{t^2}\exp\left(-\frac{d^2(r,z)}{4(1+\eta)t}\right) \tag{4.6.17}$$

where C_η is a constant which depends on $\eta > 0$.
Then by combining (4.6.16) and (4.6.17), one gets the better estimate:

Proposition 4.6.30. *For all $\varepsilon > 0$, there exist two positive constants C_ε and δ_ε such that*

$$p_t(r,z) \leq C_\varepsilon \frac{e^{-\delta_\varepsilon t}}{t^2}\exp\left(-\frac{d^2(r,z)}{4(1+\varepsilon)t}\right).$$

This elegant way of obtaining this estimate was communicating to us by Laurent Saloff-Coste.
Now, let us have a look to the measure of the subriemannian balls. Consider the riemannian metric obtain by setting that (X,Y,Z) is an orthonormal frame of the tangent space in each point. Call δ_R the induced distance. By the very definition of the distances, it is clear that the subriemannian distance δ is greater than the riemannian one δ_R, then $B(g,\rho) \subset B_R(g,\rho)$ where $B_R(g,\rho)$ is the riemannian ball of center g and radius ρ and $B(g,\rho)$ the subriemannian one. Moreover, in our case the canonical riemannian volume measure is proportional to the subriemannian invariant measure μ. Note that, as we are on a left-invariant Lie group, the Ricci tensor is the same in each point and therefore bounded from below by a constant $-K$ with $K > 0$.
Therefore, for all $g \in \mathbf{SL}(2,\mathbb{R})$ and all $\rho > 0$

$$\mu(B(g,\rho)) \leq \mu(B_R(g,\rho)) \leq C_1\exp(C_2\rho)$$

for two positive constants C_1 and C_2.

4.7 From $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ to the Heisenberg group

To conclude this chapter, we show that after a convenient scaling, the heat kernels of both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ uniformly (on compact sets) converge to the heat kernel of the Heisenberg group. This scaling is related to the fact that dilation of $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ leads to the Heisenberg group (see for instance [80] for the case of the $\mathbf{SU}(2)$ group and see [37] for its extension).
From a metric point of view it is known that the Heisenberg group is the tangent cone in the Gromov-Hausdorff sense. This means that balls of radius R for a dilating distance on $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ are getting closer and closer in a certain sense of the balls of the same radius R of the Heisenberg group. For a precise statement of it, see Mitchell theorem [76] (see also [15] and [21]). Here we will see some more precise results.
First, in our setting, the dilation of $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ towards the Heisenberg group can be seen at the level of differential operators.
Indeed through the map

$$\begin{array}{rcl} \mathbb{G} & \rightarrow & \mathbb{H} \\ \exp(r(\cos\theta X + \sin\theta Y))\exp zZ & \rightarrow & (r,\theta,z) \end{array}$$

where $\mathbb{G}$ designe both $\mathbf{SU}(2)$ or $\mathbf{SL}(2,\mathbb{R})$, we can see the vector fields X, Y and Z of $\mathbf{SL}(2,\mathbb{R})$ as first order differential operators acting on smooth functions on the Heisenberg group with support included in a small enough Carnot Carathéodory ball of radius R for the case of $\mathbf{SU}(2)$ and included in the box $[0,\infty)\times[0,2\pi]\times[-\pi,\pi]$ for the case of $\mathbf{SL}(2,\mathbb{R})$.
Let us now denote by D the dilation vector field on $\mathbb{H}$ given in cylindrical coordinates by

$$D = r\frac{\partial}{\partial r} + 2z\frac{\partial}{\partial z}$$

For $c \geq 1$ we denote by $X^c_{\mathbb{G}}$, Y^c_G and $Z^c_{\mathbb{G}}$ the dilated vector fields

$$X^c_{\mathbb{G}} = \frac{1}{\sqrt{c}}e^{-\frac{1}{2}\ln cD}X_{\mathbb{G}}e^{\frac{1}{2}\ln cD},$$

$$Y^c_{\mathbb{G}} = \frac{1}{\sqrt{c}}e^{-\frac{1}{2}\ln cD}Y_{\mathbb{G}}e^{\frac{1}{2}\ln cD},$$

$$Z^c_{\mathbb{G}} = \frac{1}{c}e^{-\frac{1}{2}\ln cD}Z_{\mathbb{G}}e^{\frac{1}{2}\ln cD}.$$

In the cylindrical coordinates of the Heisenberg group, we have for the $\mathbf{SU}(2)$ case:

$$X^c_{\mathbf{SU}(2)} = \cos(\theta+\frac{2z}{c})\frac{\partial}{\partial r} - \sin(\theta+\frac{2z}{c})\left(\sqrt{c}\tanh\frac{r}{\sqrt{c}}\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{c}\tanh\frac{r}{\sqrt{c}}} - \frac{\tanh\frac{r}{\sqrt{c}}}{\sqrt{c}}\right)\frac{\partial}{\partial\theta}\right),$$

$$Y^c_{\mathbf{SU}(2)} = \sin(\theta+\frac{2z}{c})\frac{\partial}{\partial r} + \cos(\theta+\frac{2z}{c})\left(\sqrt{c}\tanh\frac{r}{\sqrt{c}}\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{c}\tanh\frac{r}{\sqrt{c}}} - \frac{\tanh\frac{r}{\sqrt{c}}}{\sqrt{c}}\right)\frac{\partial}{\partial\theta}\right),$$

$$Z^c_{\mathbf{SU}(2)} = \frac{\partial}{\partial z},$$

so that these dilated vector fields are well-defined on the box $[0,\infty)\times[0,2\pi]\times[-\sqrt{c}\pi,\sqrt{c}\pi]$; and for the $\mathbf{SL}(2,\mathbb{R})$ case:

$$X^c_{\mathbf{SL}(2,\mathbb{R})} = \cos(\theta+\frac{2z}{c})\frac{\partial}{\partial r} - \sin(\theta+\frac{2z}{c})\left(\sqrt{c}\tanh\frac{r}{\sqrt{c}}\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{c}\tanh\frac{r}{\sqrt{c}}} - \frac{\tanh\frac{r}{\sqrt{c}}}{\sqrt{c}}\right)\frac{\partial}{\partial\theta}\right),$$

$$Y^c_{\mathbf{SL}(2,\mathbb{R})} = \sin(\theta+\frac{2z}{c})\frac{\partial}{\partial r} + \cos(\theta+\frac{2z}{c})\left(\sqrt{c}\tanh\frac{r}{\sqrt{c}}\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{c}\tanh\frac{r}{\sqrt{c}}} - \frac{\tanh\frac{r}{\sqrt{c}}}{\sqrt{c}}\right)\frac{\partial}{\partial\theta}\right),$$

$$Z^c_{\mathbf{SL}(2,\mathbb{R})} = \frac{\partial}{\partial z},$$

so that these dilated vector fields are well-defined on the box $[0,\infty)\times[0,2\pi]\times[-\sqrt{c}\pi,\sqrt{c}\pi]$. Consequently, if $f:\mathbb{H}\to\mathbb{R}$ is a smooth function with compact support, we can speak of $X^c_{\mathbb{G}}f$, $Y^c_{\mathbb{G}}f$, and $Z^c_{\mathbb{G}}f$ for both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ as soon as the dilation factor c is big enough. For the dilated sublaplacian

$$\begin{aligned} L_{\mathbb{G}}{}^c &= \frac{1}{c}e^{-\frac{1}{2}\ln cD}L_{\mathbb{G}}e^{\frac{1}{2}\ln cD} \\ &= (X^c_{\mathbb{G}})^2 + (Y^c_{\mathbb{G}})^2 \end{aligned}$$

the same remarks hold true. And it reads:

$$\begin{aligned} L^c_{\mathbf{SU}(2)} &= \frac{\partial^2}{\partial r^2} + \frac{2}{\sqrt{c}}\text{cotan}\frac{2r}{\sqrt{c}}\frac{\partial}{\partial r} + c\tan^2\frac{r}{\sqrt{c}}\frac{\partial^2}{\partial z^2} \\ &+ \frac{1}{c}\left(2 + \frac{1}{\tan^2\frac{r}{\sqrt{c}}} + \tan^2\frac{r}{\sqrt{c}}\right)\frac{\partial^2}{\partial\theta^2} + 2\left(1+\tan^2\frac{2r}{\sqrt{c}}\right)\frac{\partial^2}{\partial z\partial\theta}, \end{aligned}$$

for the $\mathbf{SU}(2)$ group and

$$\begin{aligned} L^c_{\mathbf{SL}(2,\mathbb{R})} &= \frac{\partial^2}{\partial r^2} + \frac{2}{\sqrt{c}}\text{cotanh}\frac{2r}{\sqrt{c}}\frac{\partial}{\partial r} + +c\tanh^2\frac{r}{\sqrt{c}}\frac{\partial^2}{\partial z^2} \\ &+ \frac{1}{c}\left(\frac{1}{\tanh\frac{r}{\sqrt{c}}} - \tanh\frac{r}{\sqrt{c}}\right)^2\frac{\partial^2}{\partial\theta^2} + 2\left(1-\tanh^2\frac{2r}{\sqrt{c}}\right)\frac{\partial^2}{\partial z\partial\theta}, \end{aligned}$$

for the $\mathbf{SL}(2,\mathbb{R})$ group. We can also do the same for the operators Γ and Γ_2 for which the following relations hold

$$\Gamma^c_{\mathbb{G}}(f,g) = \frac{1}{c}e^{-\frac{1}{2}\ln cD}\Gamma_{\mathbb{G}}(e^{\frac{1}{2}\ln cD}f, e^{\frac{1}{2}\ln cD}g)$$

and

$$\Gamma^c_{2,\mathbb{G}}(f,g) = \frac{1}{c^2}e^{-\frac{1}{2}\ln cD}\Gamma_{2,\mathbb{G}}(e^{\frac{1}{2}\ln cD}f, e^{\frac{1}{2}\ln cD}g).$$

Remark 4.7.1. *The dilation on the Heisenberg group implies that*

$$X^c_{\mathbb{H}} = X_{\mathbb{H}},\ Y^c_{\mathbb{H}} = Y_{\mathbb{H}} \text{ and } Z^c_{\mathbb{H}} = Z_{\mathbb{H}}.$$

Therefore

$$L^c_{\mathbb{H}} = L_{\mathbb{H}},\ \Gamma^c_{\mathbb{H}} = \Gamma_{\mathbb{H}} \text{ and } \Gamma^c_{2,\mathbb{H}} = \Gamma_{2,\mathbb{H}}.$$

With these notations, the *operator* analogue of the convergence of dilated $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ to $\mathbb{H}$ is the following:

Proposition 4.7.2. *For $\mathbb{G} = \mathbf{SU}(2)$ or $\mathbf{SL}(2,\mathbb{R})$, if $f : \mathbb{H} \to \mathbb{R}$ is a smooth function with compact support, then, uniformly,*

$$\lim_{c\to+\infty} X^c_{\mathbb{G}} f = X_{\mathbb{H}} f,$$

$$\lim_{c\to\infty} Y^c_{\mathbb{G}} f = Y_{\mathbb{H}} f,$$

$$\lim_{c\to\infty} Z^c_{\mathbb{G}} f = Z_{\mathbb{H}} f,$$

and therefore the same convergence holds for:

$$\lim_{c\to\infty} L_{\mathbb{G}}{}^c f = L_{\mathbb{H}} f,$$

$$\lim_{c\to\infty} \Gamma_{\mathbb{G}}{}^c f = \Gamma_{\mathbb{H}} f$$

and

$$\lim_{c\to\infty} \Gamma_{2,\mathbb{G}}{}^c f = \Gamma_{2,\mathbb{H}} f.$$

As a corollary, we obtain the following:

Corollary 4.7.3. *For $\mathbb{G} = \mathbf{SU}(2)$ or $\mathbf{SL}(2,\mathbb{R})$, uniformly on compact sets of $\mathbb{R}_{\geq 0} \times \mathbb{R}$,*

$$\lim_{t\to 0} \frac{d_{\mathbb{G}}(\sqrt{t}r, tz)}{\sqrt{t}} = d_{\mathbb{H}}(r,z)$$

where $d_{\mathbb{H}}$ is the Carnot-Carathéodory distance of the point (r,θ,z) to the origin in $\mathbb{H}$.

As another corollary, we obtain as annonced in the introduction that:

Corollary 4.7.4. *Neither $\mathbf{SU}(2)$ and nor $\mathbf{SL}(2,\mathbb{R})$ do not satify a $CD(\rho,\infty)$ criterion.*

Proof. Let $\mathbb{G}$ be $\mathbf{SU}(2)$ or $\mathbf{SL}(2,\mathbb{R})$ Assume that, on $\mathbb{G}$, $\Gamma_2 \geq \rho\Gamma$ for some $\rho \in \mathbb{R}$. Let f be a function on $\mathbb{H}$ with compact support and let c be big enough. Then

$$\begin{aligned} \Gamma^c_2(f) &= \frac{1}{c^2} e^{-\ln cD} \Gamma_2(f_c) \\ &\geq \frac{\rho}{c^2} e^{-\ln cD} \Gamma(f_c) \\ &= \frac{\rho}{c} \Gamma^c(f) \end{aligned}$$

where $f_c = e^{\frac{1}{2}\ln cD} f$. Now letting $c \to \infty$ gives the inequality

$$\Gamma_2(f) \geq 0$$

which we know is not true for all function f on $\mathbb{H}$ with compact support. □

We saw a convergence result for the distances but we can even obtain a stronger result for the diffusions:

Proposition 4.7.5. *On both* $\mathbf{SU}(2)$ *and* $\mathbf{SL}(2,\mathbb{R})$*, uniformly on compact sets of* $\mathbb{R}_{\geq 0}\times\mathbb{R}$*,*

$$\lim_{t\to 0} t^2 p_t(\sqrt{t}r, tz) = h_1(r,z)$$

Proof. We will only do the proof in the case of $\mathbf{SU}(2)$. Indeed, the computations to prove this result are based on the explicit formula for the heat kernel p_t on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$. Thus the computations for $\mathbf{SL}(2,\mathbb{R})$ are very closed from the ones for the $\mathbf{SU}(2)$ group and therefore the proof will be omit. They are even a little simpler as we only have one integral to deal with. Let K be a compact of $\mathbb{R}_{\geq 0}\times\mathbb{R}$ and $t>0$ sufficiently small so that $(\sqrt{t}r, tz)\in[0,\frac{\pi}{2}]\times[-\pi,\pi]$ forall $(r,z)\in K$.
According to Proposition 4.4.3 we have

$$t^2 p_t(\sqrt{t}r, tz) = \frac{1}{2\pi^2}\frac{t^{3/2}}{(4\pi^{3/2}}\int_{-\infty}^{\infty} e^{-\frac{(y+itz)^2}{4t}} q_t(\cos\sqrt{t}r\cosh y)dy.$$

The idea is now to use the estimates (4.4.3) and (4.4.4) and to study the two integrals:

$$J_1(t,r,z) = \int_{\cosh y\leq \frac{1}{\cos\sqrt{t}r}} e^{-\frac{(y+itz)^2+\arccos^2(\cos\sqrt{t}r\cosh y)}{4t}} \frac{\arccos(\cos\sqrt{t}r\cosh y)}{\sqrt{1-\cos^2\sqrt{t}r\cosh^2 y}}dy$$

and

$$J_2(t,r,z) = \int_{\cosh y\geq \frac{1}{\cos\sqrt{t}r}} e^{-\frac{(y+itz)^2-\operatorname{arccosh}^2(\cos\sqrt{t}r\cosh y)}{4t}} \frac{\operatorname{arccosh}(\cos\sqrt{t}r\cosh y)}{\sqrt{\cos^2\sqrt{t}r\cosh^2 y-1}}dy$$

since as before $p_t(r,z) = \frac{1}{16\pi^2}(J_1(t,r,z)+J_2(t,r,z))$. It is easily seen that for some constant $C>0$, uniformly on K,

$$|J_1(t,r,z)|\leq Ce^{\frac{tz^2}{4}}\sqrt{t}r.$$

Therefore $J_1(t,r,z)$ goes uniformly to 0 on K.
Let us now turn to the integral $J_2(t,r,z)$ and let us show that, uniformly, $J_2(t,r,z)$ converges to $2\pi^2 h_1(r,z)$.
Let $\varepsilon>0$. Let us observe that $|e^{\frac{iyz}{2}}e^{-\frac{r^2}{4}y\text{cotanh}y}\frac{y}{\sinh y}|$ is less than ye^{-y} for big y and all r,z.
Note also that for all $1<u\leq\cosh(y/2)$,

$$e^{-(\frac{y^2-arcosh^2 u}{4t})}\frac{\text{arcosh}u}{\sqrt{u^2-1}}\leq e^{-\frac{y^2}{8t}}$$

and for all $\cosh(y/2)\leq u\leq\cosh(y)$

$$e^{-(\frac{y^2-\text{arcosh}^2 u}{4t})}\frac{\text{arcosh}u}{\sqrt{u^2-1}}\leq\frac{y/2}{\sinh y/2}.$$

The last three quantities are integrable and do not depend on r,z, so we can find $y_1>0$ so that

$$\int_{|y|\geq y_1} e^{\frac{iyz}{2}}e^{-\frac{r^2}{4}y\text{cotanh}y}\frac{y}{\sinh y}dy\leq\varepsilon.$$

and

$$\int_{|y|\geq y_1} e^{-\frac{(y+itz)^2-\operatorname{arccosh}^2(\cos\sqrt{t}r\cosh y)}{4t}}\frac{\operatorname{arccosh}(\cos\sqrt{t}r\cosh y)}{\sqrt{\cos^2\sqrt{t}r\cosh^2 y-1}}dy\leq\varepsilon.$$

Now we study the behaviour of our integrals for small y. $|e^{\frac{iyz}{2}}e^{-\frac{r^2}{4}y\text{cotanh}y}\frac{y}{\sinh y}|$ is less than 1 for small y and $e^{-\frac{(y+itz)^2-arccosh^2(\cos\sqrt{t}r\cosh y)}{4t}}\frac{\text{arccosh}(\cos\sqrt{t}r\cosh y))}{\sqrt{\cos^2\sqrt{t}r\cosh^2 y-1}}$ is less than $e^{\frac{tz^2}{4}}$ for small y. Thus, as before there exists $0<y_0$ such that

$$\int_{|y|\leq y_0} e^{\frac{iyz}{2}}e^{-\frac{r^2}{4}y\text{cotanh}y}\frac{y}{\sinh y}dy\leq\varepsilon.$$

and

$$\int_{\text{arccosh}(\frac{1}{\cos\sqrt{t}r})\leq|y|\leq y_0} e^{-\frac{(y+itz)^2-\text{arccosh}^2(\cos\sqrt{t}r\cosh y)}{4t}}\frac{\text{arccosh}(\cos\sqrt{t}r\cosh y))}{\sqrt{\cos^2\sqrt{t}r\cosh^2 y-1}}dy\leq\varepsilon.$$

Let $y_0<y<y_1$ and $0<u\leq\cosh y-1$ by the Taylor-Lagrange development formula we have the following equality

$$\text{arccosh}(\cosh y-u)=y-\frac{1}{\sinh y}u-\frac{\tilde{y}}{\sinh^{3/2}\tilde{y}}\frac{u^2}{2}.$$

for some $\tilde{y}\in]\text{arccosh}(\cosh y-u),y[$. By applying this to $\cos\sqrt{t}r\cosh y=\cosh y-tr^2\cosh y+O(t^2r^4)\cosh y$, we get

$$\text{arccosh}(\cos\sqrt{t}r\cosh y)=y-tr^2\text{cotanh}y+O(t^2r^4)(\text{cotanh}y+\cosh^2 y\frac{\tilde{y}}{\sinh^{3/2}\tilde{y}})$$

for some $\tilde{y}\in]\text{arcosh}(\cos\sqrt{t}r\cosh y),y[$. So

$$\text{arccosh}^2(\cos\sqrt{t}r\cosh y)=y^2-tr^2y\text{cotanh}y+O(t^2r^4)(y\text{cotanh}y+y\cosh^2 y\frac{\tilde{y}}{\sinh^{3/2}\tilde{y}}).$$

and

$$e^{-\frac{(y+itz)^2-\text{arccosh}^2(\cos\sqrt{t}r\cosh y)}{4t}}=e^{\frac{-iyz}{2}}e^{-\frac{r^2}{4}y\text{cotanh}y}e^{\frac{tz^2}{4}}(1+O(t^2r^4)(y\text{cotanh}y+y\cosh^2 y\frac{\tilde{y}}{\sinh^{3/2}\tilde{y}}))$$

Finally, using also Taylor Lagrange development formula at order 1 we obtain

$$\frac{\text{arccosh}(\cos\sqrt{t}r\cosh y)}{\sqrt{\cos^2\sqrt{t}r\cosh^2 y-1}}=\frac{y}{\sinh y}-\frac{tr^2}{2}\cosh(y)(\frac{1}{\sinh^2\hat{y}}+2\frac{\hat{y}\cosh\hat{y}}{\sinh^3\hat{y}})$$

for some $\hat{y}\in]\text{arcosh}(\cos\sqrt{t}r\cosh y),y[$.
So finally, we see we can pass uniformly to the limit under the integral for $y_0\leq|y|\leq y_1$ and obtain our proposition.

□

Chapter 5

Li-Yau type estimates for the heat semigroup

The estimation of heat kernel measures is a topic which had been under thorough investigation for the last thirty years at least, see [72, 36]. Among the many techniques developed for that, the famous Li-Yau parabolic inequality [72] is a very powerful tool, which relies in Riemannian geometry bounds on the gradient on heat kernels to lower bounds on the Ricci curvature. More precisely, in the simplest form, it asserts that, if E is a smooth Riemannian manifold with dimension n and non negative Ricci curvature, then if f is any positive solution of the heat equation

$$\partial_t f = \Delta f,$$

where Δ is the Laplace Beltrami operator of E, then, if $u = \log f$

$$\partial_t u \geq |\nabla u|^2 - \frac{n}{2t}.$$

This is a very precise and powerful estimate. For the model case, which is here the Euclidean space $E = \mathbb{R}^n$ and when f is the heat kernel (that is the solution of the heat equation starting at time $t = 0$ from a Dirac mass), then this inequality is in fact an equality.

From this inequality, one may easily deduce Harnack inequalities and hence precise bounds on the heat kernel.

Many generalizations of this inequality have been developed, all of them including lower bounds on the Ricci tensor. In particular, it works for a general elliptic operator L under the assumption that it satisfies a curvature-dimension inequality $CD(\rho, n)$, which is the furthermost generalization on the notion of lower bound on the Ricci curvature, see [13, 12].

In the non elliptic case, things appear to be infinitely more complicated. In particular, most of the hypoelliptic systems do not satisfy any $CD(\rho, n)$ inequality (any reasonable notion of lower bound on the Ricci tensor leads to the value $-\infty$). Nevertheless, some Li-Yau inequalities may be obtained [28].

5.1 The subelliptic Li-Yau estimates

In this section, we will obtain Li-Yau type estimates for some subelliptic heat kernels. The framework in which we will work is the one presented in Chapter 3. Moreover we will assume that the diffusion operator L satisfies also a new curvature-dimension criterion. Of course, this framework encompasses the cases of $\mathbb{H}, \mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$. Note that a generalisation of these results was done by Baudoin and Garofalo [17] where they are also able to deal with subelliptic structures of rank 2 whose codimension is bigger than 1.

The classical method of Li and Yau [72] consists in applying the maximum principle to a carefully chosen expression. Another quite different method was done by Bakry and Ledoux in [12] where they used a Γ_2 formalism. They consider a positive solution of the heat equation $\partial_t f = Lf$, and look at the expression:

$$P_s(f(t-s)\Gamma(\ln f(t-s), \ln f(t-s)))$$

defined for $0 < s < t$.
Then, they obtain through the $CD(\rho, n)$ inequality a differential inequality

$$\Phi'(s) \geq (A\Phi(s) + B)^2 + C,$$

where A, B, C are expressions which are constant in t but may depend on the function f. Then, the parabolic Li-Yau inequality is obtained as a consequence of this differential inequality.

5.1.1 A differential inequality and the subelliptic Li-Yau estimates

Here, we shall develop this method a bit further, looking at more complicated quantities like

$$P_s(f(t-s)(a(s)\Gamma(\ln f(t-s), \ln f(t-s)) + b(s)(Z\ln f(t-s))^2)),$$

and try to get some differential inequality on it. The method developed here works quite well on the simple models developed here (Heisenberg groups, $SU(2)$, $SL(2)$), but in fact work for a larger class of hypoelliptic operators. This class was already presented in the Chapter 3 and we refer to this chapter for more details. The main features are, on each space of this class, there exists a diffusion operator L on a manifold $\mathcal{M}$ which satisfies the hyptothesis of Proposition 1.3.2 and vector fields $(X_i)_{1\leq i\leq d}$, Z and X_0 such that

$$L = \sum_{i=1}^{d} X_i^2 + X_0,$$

and such the following relations hold:

$$[L, Z] = 0 \tag{5.1.1}$$

and

$$\sum_i X_i(f)[X_i, Z](f) = 0. \tag{5.1.2}$$

We will also assume that the operator L satisfies the following inequality:

$$\forall \lambda > 0, \Gamma_2(g) \geq \frac{1}{d}(Lg)^2 + 2(Zg)^2 + \left(4\rho - \frac{2}{\lambda}\right)\Gamma(g) - 2\lambda\Gamma(Zg). \tag{5.1.3}$$

As we are principally interested by the 3-dimensional case, we will only work with this criterion with $d = 2$.
This inequality (5.1.3) is a generalisation of the $CD(\rho, n)$ curvature-dimension Bakry-Emery criterion to the subelliptic case and it is easy to see that, thanks to the Cauchy-Schwarz inequality

and the expression (2.1.7) of the Γ_2, the new criterion holds on $\mathbf{SU}(2), \mathbb{H}$ and $\mathbf{SL}(2,\mathbb{R})$ with the corresponding ρ (see Section 2.1.4).
Note also that all this framework and the results presented here were generalized by baudoin and garofalo in [17] to higher dimensional cases in which the codimension may be greater than 1.
We have then the following inequality, which is our technical starting point:

Proposition 5.1.1. *Let L be a diffusion operator on a manifold $\mathcal{M}$ which satisfies the hyptothesis of Proposition 1.3.2 and such (5.1.1) and (5.1.2) hold. Let $f : \mathcal{M} \to \mathbb{R}$ be positive. Let $t > 0$, for all $x \in \mathcal{M}$ and $s \in [0,t]$, consider the expressions for $0 \le s$ leqt.*

$$\Phi_1(s) = P_s((P_{t-s}f)\Gamma(\ln P_{t-s}f))(x)$$

and

$$\Phi_2(s) = P_s((P_{t-s}f)(Z \ln P_{t-s}f)^2)(x).$$

Then, for every differentiable, non-negative and decreasing function $b : [0,t] \to \mathbb{R}$,

$$\left(-\frac{b'}{4}\Phi_1 + b\Phi_2\right)'(s) \ge -\frac{b'(s)}{4}\left(\left(\frac{b''(s)}{b'(s)} + 2\frac{b'(s)}{b(s)} + 8\rho\right) LP_tf(x) - \frac{1}{4}\left(\frac{b''(s)}{b'(s)} + 2\frac{b'(s)}{b(s)} + 8\rho\right)^2 P_tf(x)\right).$$

First we begin by the following lemma which expresses the derivatives of Φ_1 and Φ_2.

Lemma 5.1.2. *Let $f : \mathcal{M} \to \mathbb{R}$ be positive, let $t > 0$ and consider the functions $\Phi_1(s)$ and $\Phi_2(s)$ defined as above. Then for $0 \le s \le t$, we have*

$$\Phi_1'(s) = 2P_s((P_{t-s}f)\Gamma_2(\ln P_{t-s}f))(x)$$

and

$$\Phi_2'(s) = 2P_s((P_{t-s}f)\Gamma(Z \ln P_{t-s}f))(x).$$

Proof.[proof of lemma 5.1.2] We fix a positive function f and a time $t > 0$. We perform all the following computations at a given point x. First we compute the derivatives of Φ_1 and Φ_2. We have:

$$\Phi_1'(s) = 2P_s((P_{t-s}f)\Gamma_2(\ln P_{t-s}f))$$

and

$$\Phi_2'(s) = 2P_s((P_{t-s}f)\Gamma(Z \ln P_{t-s}f)).$$

The computations are simple but tedious. The computations for the first equality are exactly the same than in the elliptic case. The computations for the second equality are more involved and are based on the two crucial facts that (5.1.1) and (5.1.2) are satisfied. To do both these computations in the same time, let V be a smooth vector field and consider the functional:

$$\Phi_V(s) = P_s((g)(V \ln g)^2).$$

where $g = P_{t-s}f$. Before we do the computations, observe that if $\phi \in C^2(\mathbb{R})$ and $f, g \in C^2(\mathbb{G})$, then

$$\Gamma(\phi(f), g) = \phi'(f)\Gamma(f,g) \tag{5.1.4}$$

and

$$L(\phi(f)) = \phi'(f)Lf + \phi''(f)\Gamma(f). \tag{5.1.5}$$

In particular for $\phi = \ln$ we have

$$f\Gamma(\ln f, g) = \Gamma(f,g) \tag{5.1.6}$$

and

$$\frac{Lf}{f} = L(\ln f) + \Gamma(\ln f). \tag{5.1.7}$$

We have

$$\Phi_V'(s) = P_s\left(L\left(g\,(V\ln g)^2\right)\right) - P_s\left(Lg\,(V\ln g)^2\right) - 2P_s\left(g\,V\left(\frac{Lg}{g}\right)V(\ln g)\right).$$

We now compute:

$$\begin{aligned}
P_s\left(L\left(g\,(V\ln g)^2\right)\right) &= P_s\left(Lg\,(V\ln g)^2\right) + P_s\left(g\,L(V\ln g)^2\right) + 2P_s\left(\Gamma\left(g,(V(\ln g)^2\right)\right) \\
&= P_s\left(Lg\,(V\ln g)^2\right) + 2P_s\left(g\,V\ln g\,LV\ln g\right) \\
&\quad +2P_s\left(g\,\Gamma(V\ln g, V\ln g)\right) + 4P_s\left(V\ln g\,\Gamma(g, V\ln g)\right).
\end{aligned}$$

By taking in account (5.1.6) and (5.1.7) we obtain:

$$\begin{aligned}
\Phi_V'(s) &= 2P_s\left(Vg\,[L,V](\ln g)\right) + 2P_s\left(g\,\Gamma(V\ln g, V\ln g)\right) \\
&\quad +4P_s\left(V\ln g\,\Gamma(g, V\ln g)\right) - 2P_s\left(Vg\,V\Gamma(\ln g, \ln g)\right).
\end{aligned}$$

We now observe that

$$\begin{aligned}
V\Gamma(\ln g, \ln g) &= V\sum_{i=1}^{d} X_i(\ln g)^2 \\
&= 2\sum_{i=1}^{d} X_i(\ln g) V X_i(\ln g) \\
&= 2\Gamma(\ln g, V\ln g) + 2\sum_{i=1}^{d} X_i(\ln g)[V, X_i](\ln g).
\end{aligned}$$

Thus

$$\begin{aligned}
\Phi_V'(s) &= 2P_s(g\,V\ln g\,[L,V]\ln g) + 2P_s(g\,\Gamma(V\ln g, V\ln g)) \\
&\quad -4P_s\left(Vg\,\sum_{i=1}^{d} X_i(\ln g)[V, X_i](\ln g)\right)
\end{aligned}$$

and finally with $g = P_{t-s}f$

$$\Phi_V'(s) = 2P_s\left(g\Gamma_2^V(\ln g, \ln g)\right) - 4P_s\left(g\,V(\ln g)\,\textstyle\sum_{i=1}^{d} X_i(\ln g)[V, X_i](\ln g)\right) \tag{5.1.8}$$

where $\Gamma_2^V(f,f)$ is defined by

$$\begin{aligned}
\Gamma_2^V(f,f) &= \frac{1}{2}\left(L\left((Vf)^2\right) - 2Vf\,VLf\right) \\
&= Vf\,[L,V]f + \Gamma(Vf, Vf).
\end{aligned}$$

First if we apply (5.1.8) with $V = X_j$ and sum in $j = 1, \dots, d$ (note that $d = 2$) then we get

$$\Phi_1'(s) = 2P_s \left(P_{t-s}f\Gamma_2(\ln P_{t-s}f)\right).$$

Indeed, by the skew-symmetry of $[X_i, X_j]$, one has:

$$\sum_{j=1}^{d} X_j g \sum_{i=1}^{d} X_i(\ln g)[X_j, X_i](\ln g) = 0.$$

Next if we aply (5.1.8) with $V = Z$ which satisfies the skew-symmetric condition (5.1.2) then we get,

$$\Phi_2'(s) = 2P_s \left(P_{t-s}f\Gamma_2^Z(\ln P_{t-s}f)\right).$$

Eventually, note that our previous computation shows also that the condition (5.1.1) $[L, Z] = 0$ implies

$$\Gamma_2^Z(f, f) = \Gamma(Zf, Zf)$$

for all smooth functions f. □

We now come to the proof of Proposition 5.1.1.

Proof.[proof of Proposition 5.1.1] With the above notation, using the new curvature-dimension criterion (5.1.3), which sets that for every $\lambda > 0$, and every smooth function g,

$$\Gamma_2(g) \geq \frac{1}{2}(Lg)^2 + 2(Zg)^2 + \left(4\rho - \frac{2}{\lambda}\right)\Gamma(g) - 2\lambda\Gamma(Zg)$$

and the Lemma 5.1.2, we obtain the following differential inequality

$$\Phi_1'(s) \geq P_s((P_{t-s}f)(L\ln P_{t-s}f)^2) + 4\Phi_2(s) + \left(8\rho - \frac{4}{\lambda}\right)\Phi_1(s) - 2\lambda\Phi_2'(s).$$

We now have that for every $\gamma \in \mathbb{R}$,

$$(L\ln P_{t-s}f)^2 \geq 2\gamma L\ln P_{t-s}f - \gamma^2,$$

and

$$L\ln P_{t-s}f = \frac{LP_{t-s}f}{P_{t-s}f} - \frac{\Gamma(P_{t-s}f)}{(P_{t-s}f)^2}.$$

Thus, for every $\lambda > 0$ and every $\gamma \in \mathbb{R}$,

$$\Phi_1'(s) \geq \left(8\rho - \frac{4}{\lambda} - 2\gamma\right)\Phi_1(s) + 4\Phi_2(s) - 2\lambda\Phi_2'(s) + 2\gamma LP_tf - \gamma^2P_tf.$$

Now for two non negative functions a and b defined on the time interval $[0, t)$, we have

$$(a\Phi_1 + b\Phi_2)' \geq \left(a' + (8\rho - \frac{4}{\lambda} - 2\gamma)a\right)\Phi_1 + (4a + b')\Phi_2 + (-2a\lambda + b)\Phi_2' + 2a\gamma LP_tf - a\gamma^2P_tf. \tag{5.1.9}$$

So, if b is a positive decreasing function on the time interval $[0, t)$, by choosing in the previous inequality

$$a = -\frac{b'}{4},$$

$$\lambda = -\frac{b}{2b'},$$

and

$$\gamma = \frac{1}{2}\left(\frac{b''}{b'} + 2\frac{b'}{b} + 8\rho\right),$$

we obtain with these notations

$$(a\Phi_1 + b\Phi_2)' \geq 2a\gamma LP_t f - a\gamma^2 P_t f$$

get the desired result. □

If we choose a function b non-increasing and such that $b(t) = b'(t) = 0$, by integrating the previous inequality between 0 and t, one gets:

$$\Gamma(\ln P_t f) + \frac{-4b(0)}{b'(0)} Z(\ln P_t f)^2 \leq A(t)\frac{LP_t f}{P_t f} + B(t) \tag{5.1.10}$$

with

$$A(t) = \int_0^t \frac{b'(s)}{b'(0)}(-\gamma(s))ds,$$

$$B(t) = \frac{1}{4}\int_0^t \frac{b'(s)}{b'(0)}\gamma(s)^2 ds$$

and

$$\gamma(s) = \frac{b''}{b'} + 2\frac{b'}{b} + 8\rho.$$

Note that $\frac{LP_t f}{P_t f} = \partial_t \ln(P_t f)$.
Let us now specify the inegality (5.1.10) for some particular choices of functions b. First, the choice $b(s) = (t-s)^\alpha$ with $\alpha > 2$ gives

Theorem 5.1.3. *For all $\alpha > 2$, for every positive function f and $t > 0$,*

$$\Gamma(\ln P_t f) + \frac{t}{\alpha}(Z \ln P_t f)^2 \leq \left(\frac{3\alpha - 1}{\alpha - 1} - \frac{2\rho t}{\alpha}\right)\frac{LP_t f}{P_t f} + \frac{\rho^2 t}{\alpha} - \frac{\rho(3\alpha - 1)}{\alpha - 1} + \frac{(3\alpha - 1)^2}{\alpha - 2}\frac{1}{t} \tag{5.1.11}$$

Let us give a few remarks for this result. First, for $\rho = 0$, it has a simpler form. And as the inequality (5.1.3) for $\rho \geq 0$ implies the same one with $\rho = 0$, the inequality (5.1.11) with $\rho = 0$ is valid for $\rho \geq 0$.

Corollary 5.1.4. *On $\mathbb{H}$ and $\mathbf{SU}(2)$, there exist constants A, B and C such that,*

$$\Gamma(\ln P_t f) + Ct(Z \ln P_t f)^2 \leq A\partial_t \ln P_t f + \frac{B}{t}.$$

Moreover, one can choose $A = \frac{3\alpha-1}{\alpha-1}$, $B = \frac{(3\alpha-1)^2}{\alpha-2}$ and $C = \frac{1}{\alpha}$ for each choice of $\alpha > 2$.

In particular, with $D = \frac{B}{A}$, one gets $\partial_t \ln P_t f \geq -\frac{D}{t}$, which gives

$$P_t f \leq t^{-B/A} P_1 f.$$

On the Heisenberg group, one sees that the behavior of $P_t f$ when t goes to 0 is of order t^{-2} (a simple dilation argument shows that). Therefore, one sees that the optimal constant D in the previous inequality is $D = 2$. Unfortunately, it can be shown by some elementary considerations that the best constant one may obtain from the previous proposition shall always produce a constant $D > 2$. This is a strong difference with the classical parabolic Li-Yau inequality where the inequality

$$\partial_t u \geq -\frac{n}{2t}$$

gives the right order of magnitude of the heat kernel near $t = 0$.
Now when $\rho > 0$, we easily get an exponential decay by using the function:

$$b(s) = \left(e^{-\frac{8\rho s}{3\alpha}} - e^{-\frac{8\rho t}{3\alpha}} \right)^{\alpha}, \quad \alpha > 2.$$

This writes:

Corollary 5.1.5. *For every $\rho > 0$ and every $\alpha > 2$, for every positive function f, $x \in \mathbf{G}$ and $t > 0$,*

$$\Gamma(\ln P_t f) + \frac{3}{2}\left(1 - e^{-\frac{8\rho t}{3\alpha}}\right)(Z \ln P_t f)^2 \leq 3\,(3\alpha - 1)\,\frac{\alpha}{\alpha - 1} e^{-\frac{8\rho t}{3\alpha}} \frac{LP_t f}{P_t f} + 6\rho \frac{(3\alpha - 1)^2}{\alpha(\alpha - 2)} \frac{e^{-\frac{16\rho t}{3\alpha}}}{1 - e^{-\frac{8\rho t}{3\alpha}}} \tag{5.1.12}$$

The idea why we take the function $b(s) = \left(e^{-\frac{8\rho s}{3\alpha}} - e^{-\frac{8\rho t}{3\alpha}} \right)^{\alpha}$ is the following: we search a function b such that we are able to compute explicitely the quantities $A(t)$ and $B(t)$. So we to try to solve the differential equation:

$$\frac{b''}{b'} + 2\frac{b'}{b} + 8\rho = cb^{\beta}.$$

The change of functions $u = b^3$ gives:

$$\frac{u''}{u'} + 8\rho = cu^{\frac{\beta}{3}}$$

and so by integrating

$$u' + 8\rho u = c_1 u^{\frac{\beta}{3}+1} + c_2.$$

If we choose $c_2 = 0$, we recognize a Bernoulli differential equation:

$$\frac{u'}{u^{\frac{\beta}{3}+1}} + \frac{8\rho}{u^{\frac{\beta}{3}}} = c_1,$$

Then the change of function $v = \frac{1}{u^{\frac{\beta}{3}}}$ gives:

$$v' - \frac{8\rho\beta}{3} v = c_3.$$

Therefore

$$v = C_1 \exp(\frac{8\rho\beta}{3} s) + C_2,$$

$$u = \left(C_1 \exp(\frac{8\rho\beta s}{3}) + C_2 \right)^{-\frac{3}{\beta}}$$

and eventually with setting $\alpha = -\frac{1}{\beta}$

$$b(s) = \left(C_1 \exp(-\frac{8\rho s}{3\alpha}) + C_2 \right)^{\alpha}.$$

Now the choice of the initial conditions is coming from the conditions $b(t) = b'(t) = 0$, implies $\alpha > 2$ and leads to our choice of the function $b(s) = \left(e^{-\frac{8\rho s}{3\alpha}} - e^{-\frac{8\rho t}{3\alpha}} \right)^{\alpha}$.
In fact the inequality (5.1.12) is also true for $\rho < 0$ but it is clearly weaker than (5.1.11).

5.1.2 A more careful analysis of the differential inequality

Now we will do a more precise study of the differential inequality of Theorem 5.1.1. This will enable us to recover the compactness of the space in the case $\rho > 0$. Of course we know it for the $\mathbf{SU}(2)$ case. But which is interesting (and what we will prove in fact) is that if a 3-dimensionnal subelliptic operator satisfy the hypothesis of Proposition 1.3.2, the antisymmetric conditions (5.1.1), (5.1.2) and the curvature-dimension condition (5.1.3) with $\rho > 0$, it is a compact space. Note also that we can also easily extend this result when the dimension d is greater than 2.
In the case $\rho \le 0$, this study will give us the best order we can expext for the constants $A(t)$ and $B(t)$ in the Li-Yau inequality (5.1.10).

Start with this inequality and set $V(b) = -b^2 b'$ for b a positive decreasing function such that $b(t) = b'(t) = 0$. The constraints that the function V on $[0, b_0]$ must satisfy are

$$V(x) > 0 \text{ for } x > 0, \tag{5.1.13}$$

$$t = \int_0^{b_0} \frac{x^2}{V(x)} dx \tag{5.1.14}$$

and

$$\left(\frac{V(x)}{x^2} \right)_{x=0} = 0. \tag{5.1.15}$$

The first one is just coming from the fact that b is a positive decreasing function. The second one is coming from

$$t = -\int_0^t \frac{b^2(s)}{V(b(s))} b'(s) ds$$

and the change of variables $x = b(s)$, recalling $b(t) = 0$:
The third one is coming from the fact that both $b(t)$ and $b'(t)$ equal 0.
As $V(b) = -b^2 b'$, note that

$$V'(b)b' = -b^2 b'' - 2bb'^2$$

and so

$$\frac{V'(b)}{b^2} = -\left(\frac{b''}{b'} + 2\frac{b'}{b} \right).$$

We then get with $u = \ln P_t f$ and $a_0 = \frac{V(b_0)}{b_0^2}$

$$a_0 \Gamma(u) + b_0 (Z u_t)^2 \le A \partial_t u + B,$$

where for any choice of such a function V, one has

$$\begin{aligned} A &= \int_0^t \frac{b'(s)}{4}\left(\frac{b''(s)}{b'(s)}+2\frac{b'(s)}{b(s)}+8\rho\right)ds \\ &= -\int_0^t \frac{b'(s)}{4}\left(\frac{V'(b)}{b^2}-8\rho\right)ds \\ &= \frac{1}{4}\int_0^{b_0}\left(\frac{V'(x)}{x^2}-8\rho\right)dx \end{aligned}$$

and

$$\begin{aligned} B &= -\int_0^t \frac{b'(s)}{16}\left(\frac{b''(s)}{b'(s)}+2\frac{b'(s)}{b(s)}+8\rho\right)^2 ds \\ &= \frac{1}{16}\int_0^{b_0}\left(\frac{V'(x)}{x^2}-8\rho\right)^2 dx. \end{aligned}$$

In this system, we see that if we change both $\tilde{b}=\frac{b}{\lambda}$ and $\tilde{V}(x)=\frac{V(\lambda x)}{\lambda^3}$, the two constraints (5.1.14) and (5.1.15) are still satisfied for $\tilde{b}$ and $\tilde{V}$. This change multiply every constant a_0, A and B by $\frac{1}{\lambda}$. Therefore, we may assume that $b_0=1$ without any loss and get

$$V(1)\Gamma(u)+(Zu)^2 \le \frac{1}{4}\int_0^1\left(\frac{V'(x)}{x^2}-8\rho\right)dx\partial_t u+\frac{1}{16}\int_0^1\left(\frac{V'(x)}{x^2}-8\rho\right)^2 dx.$$

Also, taking $W(s)=tV(s)$ allows us to reduce to the case $t=1$. So finally we have rephrased the problem as follows. For any non negative function V on $[0,1]$ such that

$$\int_0^1 \frac{x^2}{V(x)}dx=1,\ \left(\frac{V(x)}{x^2}\right)_{x=0}=0,$$

and for any $u=\log P_t f$ with $f\ge 0$ one has

$$\Gamma(u)+\frac{t}{V(1)}(Zu)^2 \le \frac{\alpha(V)-8\rho t}{V(1)}\partial_t u+\frac{1}{4t}\frac{\beta(V)-\alpha^2(V)+(\alpha(V)-8\rho t)^2}{V(1)} \tag{5.1.16}$$

where

$$\alpha(V)=\int_0^1\frac{V'(x)}{x^2}dx,\ \beta(V)=\int_0^1\left(\frac{V'(x)}{x^2}\right)^2 dx.$$

For the constant $\alpha(V)$ and $\beta(V)$ one has the following inequalities

$$\alpha(V)\ge V(1)+8 \tag{5.1.17}$$

and

$$\beta(V)>\alpha(V)^2. \tag{5.1.18}$$

The first inequality comes from an integration by parts

$$\int_0^1\frac{V'(x)}{x^2}dx=V(1)-\left(\frac{V(x)}{x^2}\right)_{x=0}+2\int_0^1\frac{V}{x^3}dx,$$

and the Cauchy-Schwarz inequality

$$\int_0^1 \frac{V(x)}{x^3}dx \int_0^1 \frac{x^2}{V(x)}dx \geq \left(\int_0^1 \sqrt{\frac{V(x)}{x^3}}\sqrt{\frac{x^2}{V(x)}}dx\right)^2 = \left(\int_0^1 \frac{dx}{\sqrt{x}}\right)^2 = 4.$$

Note it implies that $\alpha(V)$ is positive.
The second one is coming directly from the Cauchy-Schwarz inequality. It is interesting to note that no equality can occur in this inequality. Indeed, it would be the case only for $\frac{V'(x)}{x^2}$ constant that is $V(x) = cx^3$, but the constraint (5.1.14) can not be satisfied for such a function since the integral which appeared there is infinite.
The preceding calculus is valid for any ρ.
We begin by studying the case $\rho = 0$. We have:

$$A(t) = \frac{\alpha(V)}{V(1)},$$

$$B(t) = \frac{1}{4t}\frac{\beta(V)}{V(1)}$$

and

$$\frac{B(t)}{A(t)} = \frac{1}{4t}\frac{\beta(V)}{\alpha(V)}.$$

Therefore, by the inequalities (5.1.17) and (5.1.18)

$$\begin{aligned} \frac{B(t)}{A(t)} &= \frac{1}{4t}\alpha(V)\frac{\beta(V)}{\alpha^2(V)} \\ &> \frac{1}{4t}(V(1)+8)\frac{\beta(V)}{\alpha^2(V)} \\ &> \frac{2}{t}. \end{aligned}$$

But it is not possible to choose a function V such that both $\frac{\beta(V)}{\alpha^2(V)}$ tends to 1 and $V(1)$ tends to 0. Therefore it is not possible to be close to the optimal inequality as discussed before.

Let us now have a look to the case $\rho < 0$. By the inequalities (5.1.17) and (5.1.18), we have:

$$A(t) = \frac{\alpha(V) + 8|\rho|t}{V(1)} > 1,$$

$$B(t) = \frac{1}{4t}\frac{\beta(V) - \alpha^2(V) + (\alpha(V) + 8|\rho|t)^2}{V(1)} > \frac{5}{4t} + 4,$$

and

$$\frac{B(t)}{A(t)} \geq \frac{1}{4t}\frac{\alpha(V) + 8|\rho|t}{V(1)} \geq 4|\rho|.$$

It is then easy to see that the function $b(s) = (t-s)^t$ gives the right orders of this constant (just take $\alpha = t$ in theorem 5.1.3). Note that the function $b(s) = (t-s)^\alpha$ corresponds to the function $V(x) = \lambda x^\beta$ for some constants λ and β. Let us summarize the results for $\rho < 0$. $A(t)$ and $B(t)$ are always positive. For t small, one can have $A(t)$ of the order of a constant and $B(t)$ of order of $\frac{C}{t}$. For t big, one can have both $A(t)$ and $B(t)$ of the order of a constant. Therefore in small time the constants $A(t)$ and $B(t)$ behave in the same way as the ones on $\mathbb{H}$. This is not anymore the case in big times.

5.1.3 The Myers diameter theorem

In this section, we will see that in the case $\rho > 0$, we can obtain a compacity theorem like the one of Myers in Riemannian geometry. Note that another compacity theorem was proved by Rumin [82] for general 3-dimensional subelliptic operators, that is without the condition on the tosion. Observe that in this case $\rho > 0$ the term $\alpha(V) - 8\rho t$ can be made negative, and therefore we may get as in the elliptic case with strictly positive Ricci bound a universal upper bound on $|\partial_t u|$.

Corollary 5.1.6. *Let us assume $\rho > 0$. There exist $t_0 > 0$ and $C > 0$, such that for any positive function f,*

$$|\partial_t \ln P_t f(x)| \leq C \exp\left(-\frac{4\rho t}{3}\right), \quad t \geq t_0, x \in \mathbf{G}.$$

Proof. To make the term $\beta(V) - \alpha^2(V)$ small we are lead to choose $V(x) = \lambda x^3$ on $[\epsilon, 1]$ and $V = \lambda \epsilon^{3-\gamma} x^\gamma$ on $[0, \epsilon]$, for some fixed $\gamma \in (5/2, 3)$. The constraint on V implies

$$\lambda = -\log \epsilon + \frac{1}{3-\gamma}.$$

Meanwhile, we have

$$\alpha = \lambda \left(3 + 2\epsilon \frac{3-\gamma}{\gamma-2}\right),$$

and

$$\beta = \lambda^2 \left(9 + \epsilon \frac{(15-\gamma)(3-\gamma)}{2\gamma-5}\right),$$

so that

$$\beta - \alpha^2 = \lambda^2 \epsilon \frac{(3-\gamma)^2}{\gamma-2} \left(\frac{\gamma+10}{2\gamma-5} + \epsilon \frac{4}{\gamma-2}\right).$$

By taking

$$\epsilon = \exp\left(-\frac{8\rho}{3} t + \frac{1}{3-\gamma} + R\right)$$

for t large enough to ensure $\varepsilon < 1$, one obtains

$$\alpha - 8\rho t \simeq -3R$$

and

$$\beta - \alpha^2 \simeq C t^2 \varepsilon \simeq C t^2 \exp\left(-\frac{2\rho t}{3}\right).$$

With $R = ct \exp(-\frac{4\rho t}{3})$ the terms $(\alpha - 8\rho t)^2$ and $\beta - \alpha^2$ are of the same order and playing now with the sign of c, one gets

$$|\partial_t u| \leq C \exp\left(-\frac{4\rho t}{3}\right).$$

□

Proposition 5.1.7. *Let us assume $\rho > 0$. The spectrum of $-L$ lies in $\{0\} \cup [\frac{4\rho}{3}, +\infty]$.*

Proof. We fix $x \in \mathbf{G}$ and denote by $p_t(x,\cdot)$ the heat kernel starting from x. We have for $t \geq t_0$,

$$| \partial_t \ln p_t(x,y) | \leq C \exp\left(-\frac{4\rho t}{3}\right). \tag{5.1.19}$$

The quantity $\exp\left(-\frac{\rho t}{3}\right)$ is integrable at infinity, this shows us that $\ln p_t$ converges when $t \to \infty$. Indeed for $t \geq t_0$,

$$\ln p_t(x,y) = \ln p_{t_0}(x,y) + \int_{t_0}^{t} \partial_s \ln p_s(x,y) ds.$$

Let us call $\ln p_\infty$ this limit. Moreover, from Corollary 5.1.5, $\Gamma(\ln p_t)$ is bounded above by a constant $C(t)$ which goes to 0 when t goes to ∞.
Let $u(y) = \frac{1}{\sqrt{C(t)}} \ln p_t(x,y)$, then $\Gamma(u) \leq 1$ and therefore from (1.3.1):

$$|u(y_1) - u(y_2)| \leq d(y_1, y_2)$$

for the associated Carnot-Carathéodory distance d. That is

$$|\ln p_t(x,y_1) - \ln p_t(x,y_2)| \leq \sqrt{C(t)} d(y_1,y_2).$$

In the limit, $\ln p_\infty(x,\cdot)$ is a real constant and so $p_\infty(x,\cdot)$ is a positive constant $C(x)$. In fact by the symmetry property, $p_t(x,y) = p_t(y,x)$, so that actually $C(x)$ does not depend on x.
Now as $\int p_t(x,y) d\mu(y) = 1$ for all $t > 0$, using Fatou lemma, one gets

$$C(x)\mu(G) = \int \liminf_{t \to 0} p_t(x,y) d\mu(y) \leq \liminf_{t \to 0} \int p_t(x,y) d\mu(y) = 1.$$

We deduce from this that the invariant measure μ is finite. We may then as well suppose that this measure is a probability, in which case $p_\infty = 1$. By integrating the inequality (5.1.19) from t to ∞ we therefore obtain for $t \geq t_0$:

$$| \ln p_t(x,y) | \leq C_2 \exp\left(-\frac{4\rho t}{3}\right)$$

and thus

$$\exp\left(-C_2 \exp\left(-\frac{4\rho t}{3}\right)\right) \leq p_t(x,y) \leq \exp\left(C_2 \exp\left(-\frac{4\rho t}{3}\right)\right).$$

This implies by the Cauchy-Schwarz inequality that for $f \in L^2(\mu)$ such that $\int f d\mu = 0$,

$$(P_t f)^2 \leq C_3 \exp\left(-\frac{8\rho t}{3}\right) \int f^2 d\mu.$$

For a symmetric Markov semigroup P_t, this is a standard fact (see [5] for example) that this is equivalent to say that the spectrum of $-L$ lies in $\{0\} \cup [4\rho/3, \infty)$, or equivalently that we have a spectral gap inequality: for any function f in L^2 such that ∇f is in L^2, one has

$$\int f^2 d\mu \leq \left(\int f d\mu\right)^2 + \frac{3}{4\rho} \int |\nabla f|^2 d\mu. \tag{5.1.20}$$

$\square$

Remark 5.1.8. *It can be shown that the spectral gap is actually* 2ρ *and not* $\frac{4\rho}{3}$.

We can now conclude with a substitute of the Myers's theorem:

Proposition 5.1.9. *Assume that* $\rho > 0$*, then the diameter of* L *for the Carnot-Caratheodory distance is finite.*

Proof. We are now going to prove a Sobolev inequality for the invariant measure μ. Indeed, for $0 < t \leq t_0$ we have

$$\partial_t \ln p_t \geq -\frac{C}{t},$$

from which we get by integrating between t and t_0

$$\ln p_{t_0} - \ln p_t \geq -C \ln(t_0/t),$$

and therefore

$$\ln p_t \leq A - C \log t$$

with $A = || \ln p_t||_\infty + C \ln t_0$. This gives the ultracontractivity of the semigroup P_t with a polynomial bound t^{-C} when $t \to 0$:

$$||p_t||_\infty \leq \frac{e^A}{t^C}.$$

Now it is a well known fact (see [90, 5]) that this last property is equivalent to a Sobolev inequality

$$\left(\int f^{\frac{2C}{C-1}} d\mu \right)^{\frac{C-1}{C}} \leq A \int f^2 d\mu + B \int \|\nabla f\|^2 d\mu. \tag{5.1.21}$$

When we have both Sobolev inequality (5.1.21) and spectral gap inequality (5.1.20) then (see [5]) we have a tight Sobolev inequality, that is the Sobolev inequality (5.1.21) with $A = 1$.
In this situation, the diameter of E with respect to the distance defined in 1.3.1 is finite (see [11]), which concludes the proof.

□

Remark 5.1.10. *This proof of the finitness of the diameter only uses the above Li-Yau estimates: if a metric space satisfies these estimates with* $\rho > 0$*, then it has a finite diameter and therefore is compact if moreover the metric is complete.*
However, this proof, contrary to the Riemannian case, does not give any bounds on the diameter. In the Riemannian case, the condition $CD(\rho, n)$ *with* $\rho > 0$ *implies, by non linear analysis, the optimal Sobolev inequality. This step is done by showing the existence of an extremal function* f *in the Sobolev inequality and by making the changes of function* f^s *and* f^r *in both the equation for the extremal function and the* $CD(\rho, n)$ *criterion. It does not seem easy to generalize this step in our subelliptic setting. Here to obtain the Sobolev inequality, we use the ultracontractivity but we do not have any bounds on the constants. Now, in the Riemannian case, from the optimal Sobolev inequality, also by non-linear methods, one can obtain the optimal diameter bound. For some references on this subject, one can consult [11] and [44].*

Actually it is possible to obtain an explicit (of course not sharp) bound for the diameter using only linear methods. Let us do it. Let $\rho > 0$. As a corollary of the inequality (5.1.12), one obtains the following upper bound for the heat kernel.

Corollary 5.1.11. *Let $\rho > 0$, then for all $(r, z) \in \mathbb{G}$ and all $t > 0$:*

$$p_t(r,z) \leq \frac{1}{(1-e^{-\gamma t})^{\frac{D}{2}}} \tag{5.1.22}$$

with

$$\gamma = \frac{8\rho}{3\alpha} \text{ and } D = \frac{3}{2}\frac{(\alpha-1)(3\alpha-1)}{\alpha(\alpha-2)}$$

for all $\alpha > 2$.

Because we know our result is not optimal, we do not try to optimize in $\alpha > 2$ the last quantity.

Proof. Aplying (5.1.12) to the heat kernel itself, one obtains that for all $t > 0$ and all $\alpha > 2$:

$$\partial_t(\ln p_t) \geq -2\frac{\rho}{\alpha}\frac{(\alpha-1)(3\alpha-1)}{\alpha(\alpha-2)}\frac{e^{-\frac{8\rho}{3\alpha}t}}{1-e^{-\frac{8\rho}{3\alpha}t}}.$$

Integrating the last inequality between t and ∞ gives

$$-\ln p_t \geq \frac{3}{4}\frac{(\alpha-1)(3\alpha-1)}{\alpha(\alpha-2)}\ln\left(1-e^{-\frac{8\rho}{3\alpha}t}\right)$$

from which the conclusion easily follows. $\square$

The idea is now to see that the ultracontractive bound implies (5.1.22) that the operator L satisfies an entropy-energy inequality. Such inequalities have been extensively studied by Bakry in [5] (see chapter 4 and 5).

Proposition 5.1.12. *Let $\rho > 0$. With the previous notations, for $f \in L^2(\mathbb{G})$, we have*

$$\int_{\mathbb{G}} f^2 \ln f^2 d\mu - \int_{\mathbb{G}} f^2 d\mu \ln\left(\int_{\mathbb{G}} f^2 d\mu\right) \leq \Phi\left(\int_{\mathbb{G}} \Gamma(f) d\mu\right)$$

where

$$\Phi(x) = D\left[\left(1+\frac{2}{\gamma D}\right)\ln\left(1+\frac{2}{\gamma D}\right) - \frac{2}{\gamma D}x\ln\left(\frac{2}{\gamma D}x\right)\right].$$

Proof. From corollary (5.1.11), for every $f \in L^2(\mathbb{G})$, since μ has a finite mass,

$$\|P_t f\|_\infty \leq \frac{1}{(1-e^{-\gamma t})^{\frac{D}{2}}}\|f\|_2.$$

Therefore, from Davies theorem (Theorem 2.2.3 in [36]), it gives the following logarithmic Sobolev inequality:

$$\int_{\mathbb{G}} f^2 \ln f^2 d\mu - \int_{\mathbb{G}} f^2 d\mu \ln\left(\int_{\mathbb{G}} f^2 d\mu\right) \leq 2t\int_{\mathbb{G}} \Gamma(f) d\mu - D\ln(1-e^{-\gamma t}),\ t > 0.$$

Now, we may try to optimize the right hand side term in t. The minimum is obtained for

$$t = -\frac{1}{\gamma}\ln\left(\frac{2x}{\gamma D + 2x}\right)$$

with $x = \int_{\mathbb{G}} \Gamma(f) d\mu$ and its value is then

$$\begin{aligned}
&\left(\frac{-2x}{\gamma}\right) \ln \left(\frac{2x}{\gamma D + 2x}\right) - D \ln \left(1 - \frac{2x}{\gamma D + 2x}\right) \\
=&\frac{-2x}{\gamma} \ln \left(\frac{2x}{\gamma D}\right) + \frac{2x}{\gamma} \ln \left(\frac{\gamma D + 2x}{\gamma D}\right) + D \ln \left(\frac{\gamma D + 2x}{\gamma D}\right) \\
=&D \left[\left(1 + \frac{2}{\gamma D} x\right) \ln \left(1 + \frac{2}{\gamma D} x\right) - \frac{2}{\gamma D} x \ln \left(\frac{2}{\gamma D} x\right) \right].
\end{aligned}$$

□

With Proposition 5.1.12, we can now give an explicit bound for the diameter.

Proposition 5.1.13. *Under the previous hyptohesis, the diameter of* $\mathbb{G}$ *satisfies the bound*

$$\mathit{diam}\ \mathbb{G} \le 2\sqrt{2} \sqrt{\frac{D}{\gamma}} \pi.$$

Proof. The function Φ that appears in Proposition 5.1.12 enjoys the folloing properties:

- Φ is non decreasing
- $\Phi(0) = 0$
- Φ is concave
- $\frac{\Phi'(x)}{\sqrt{x}}$ is integrable in the neighborhoods of 0 and of ∞.

Indeed,

$$\Phi'(x) = \frac{2}{\gamma D} \ln \left(1 + \frac{\gamma D}{2x}\right).$$

Therefore, one can apply the Theorem 5.4 in [5] to deduce that the diameter of $\mathbb{G}$ is finite and

$$\text{diam } \mathbb{G} \le \int_0^\infty \frac{\Phi'(x)}{\sqrt{x}} dx < \infty.$$

By an integration by parts and a routine computation, the last quantity equals

$$\begin{aligned}
\int_0^\infty \frac{\Phi'(x)}{\sqrt{x}} dx &= \int_0^\infty \frac{4D}{\sqrt{x}(2x + \gamma D)} dx \\
&= 2\sqrt{2} \sqrt{\frac{D}{\gamma}} \pi.
\end{aligned}$$

□

Remark 5.1.14. *One can explicit the constant:*

$$2\sqrt{2} \sqrt{\frac{D}{\gamma}} \pi = \frac{3}{\sqrt{2}} \sqrt{\frac{(\alpha - 1)(3\alpha - 1)}{(\alpha - 2)}} \frac{1}{\sqrt{\rho}}.$$

The minimum in $\alpha > 2$ *of this quantity is obtain for* $\alpha \simeq 3.29$*. For* $\alpha = 3$*, it equals* $6\sqrt{2}\frac{1}{\sqrt{\rho}}$*. Of course as we said it before, there is no hope to obtain the optimal constant by this method.*

5.2 Other consequences of the Li-Yau type estimates

As in the riemannian case, these Li-Yau type estimates have a lot of consequences in term of functional inequalities.
In this section, we will first write the consequences of our Li-Yau estimates for the general case and then specialize the results to each model space. The general case of the Li-Yau type estimates is the following

$$\Gamma(\ln P_t f) \leq A(t)\frac{LP_t f}{P_t f} + B(t) \tag{5.2.23}$$

with A and B positive functions. It turns out, as we saw it, that in the case $\rho > 0$ it is possible to choose a good function b such that (5.2.23) holds true with A negative. But here for simplicity in the exposition we work only with A positive. It is clear that B has to be positive.

5.2.1 Harnack inequality

To obtain a Harnack inequality as in the classical case, we integrate along the geodesic and use (5.2.23). Let $u(t,x)$ be a positive solution of the heat equation. Let x_1, x_2 be two points of G and $0 < t_1 < t_2$, consider the (or rather a for points in the cut locus) minimising geodesic γ with constant speed between x_1 and x_2 such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. Its speed is then $\frac{d(x_1,x_2)}{t_2-t_1}$.

$$\begin{aligned}
\ln\frac{u(x_2,t_2)}{u(x_1,t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt}\ln u(\gamma(t),t)dt \\
&= \int_{t_1}^{t_2} \partial_t(\ln u) + < \nabla \ln u, \frac{d\gamma}{dt} > dt.
\end{aligned}$$

By (5.2.23) we have

$$\partial_t(\ln u) \geq \frac{1}{A(t)}\Gamma(\ln u) - \frac{B(t)}{A(t)}.$$

Moreover by using Cauchy-Schwarz inequality, for $\lambda > 0$, the following inequality holds:

$$< \nabla \ln u, \frac{d\gamma}{dt} > \geq -\frac{1}{2\lambda}\Gamma(\ln u) - \frac{\lambda}{2}\frac{d^2(x_1,x_2)}{(t_2-t_1)^2}.$$

The choice $\lambda = \frac{A(t)}{2}$ gives:

$$\ln\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq -\frac{d^2(x_1,x_2)\int_{t_1}^{t_2} A(t)dt}{4(t_2-t_1)^2} - \int_{t_1}^{t_2}\frac{B(t)}{A(t)}dt$$

and finally:

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \exp\left(\int_{t_1}^{t_2}\frac{B(t)}{A(t)}dt\right)\exp\left(\frac{d^2(x_1,x_2)\int_{t_1}^{t_2} A(t)dt}{4(t_2-t_1)^2}\right). \tag{5.2.24}$$

Now we can set the Harnack inequality on each of the three model spaces using the previous computations of $A(t)$ and $B(t)$. On $\mathbb{H}$, the Li-Yau inequality of Corollary (5.1.4), we get the following Harnack inequality:

Proposition 5.2.1. *There exist two positive constants A_1 and A_2 such that for $0 < t_1 < t_2$ and $g_1, g_2 \in \mathbb{H}$*

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{A_1} \exp\left(A_2 \frac{\delta_{\mathbb{H}}(g_1, g_2)^2}{t_2 - t_1}\right) \tag{5.2.25}$$

On $\mathbf{SU}(2)$, since the heat kernel p_t converges towards 1 when t goes to infinity, the Harnack inequality is only really interesting for t small. However we can set it for all $t > 0$ using the Li-Yau inequality of Corollary (5.1.4) with $\rho = 0$.

Proposition 5.2.2. *There exist two positive constants A_1 and A_2 such that for $0 < t_1 < t_2 \leq 1$ and $g_1, g_2 \in \mathbf{SU}(2)$*

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{A_1} \exp\left(A_2 \frac{\delta_{\mathbf{SU}(2)}(g_1, g_2)^2}{t_2 - t_1}\right). \tag{5.2.26}$$

Remark 5.2.3. *The constants A_1 and A_2 are the same on $\mathbb{H}$ and $\mathbf{SU}(2)$ since we use the same Li-Yau inequality and one can take*

$$A_1 = \frac{(3\alpha - 1)(\alpha - 1)}{\alpha - 2} \text{ and } A_2 = \frac{3\alpha - 1}{4(\alpha - 1)}$$

for each choice of $\alpha > 2$.

On $\mathbf{SL}(2, \mathbb{R})$, the Harnack inequality writes:

Proposition 5.2.4. *There exist two positive constants B_1 and B_2 such that for $0 < t_1 < t_2 \leq 1$ and $g_1, g_2 \in \mathbf{SL}(2, \mathbb{R})$*

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{B_1} \exp\left(B_2 \frac{\delta_{\mathbf{SL}(2,\mathbb{R})}(g_1, g_2)^2}{t_2 - t_1}\right) \tag{5.2.27}$$

and there exists two positive constants $\tilde{B}_1$ and $\tilde{B}_2$ such that for $2 < t_1 < t_2$ and $g_1, g_2 \in \mathbf{SL}(2, \mathbb{R})$

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \exp\left(\tilde{B}_1(t_2 - t_1)\right) \exp\left(\tilde{B}_2 \frac{\delta_{\mathbf{SL}(2,\mathbb{R})}(g_1, g_2)^2}{t_2 - t_1}\right). \tag{5.2.28}$$

5.2.2 Ultracontractivity

Now we can also use methods of Davies [36] to obtain some ultracontractivity for the heat kernel. We work first on $\mathbb{H}$ and $\mathbf{SU}(2)$ since they satisfy the same Harnack inequality valid for all times $t > 0$. Now, let $t > 0$, since $A(t) \leq C_1$ and $\frac{B(t)}{A(t)} \leq \frac{C_2}{t}$, then by applying (5.2.24) for the heat kernel with $t_1 = t$, $t_2 = 2t$ and $d(x_1, x_2) \leq R$:

$$p_t(x_1) \leq \exp C_2 \exp\left(\frac{C_1 R^2}{t}\right) p_{2t}(x_2).$$

By chosing $R = \sqrt{t}$ and integrating in the variable x_2 over the ball $B(x_1, R)$, one gets

$$\mu(B(x_1, \sqrt{t})) p_t(x_1) \leq \exp C_1 \exp C_2$$

and therefore:

$$p_t(x) \leq \frac{C}{\mu(B(0,\sqrt{t}))}$$

since on left-invariant Lie groups the volume of a ball does not depend on the center of the ball. Now the measure of $B(0,R)$ is of order R^4 for all $R>0$ for the Heisenberg group, and we obtain,

$$p_t(x) \leq \frac{C'}{t^2}, \text{ on } \mathbb{H} \tag{5.2.29}$$

for all $x \in \mathbb{H}$ and all $t>0$.
In fact, by the ball-box theorem (see [77]), it is known that on $\mathbf{SU}(2)$, for R small, $\mu(B(x_1,R))$ is of order R^4. We therefore obtain the following ultracontractive bound in small times on $\mathbf{SU}(2)$.

$$p_t(x) \leq \frac{C'}{t^2}, \text{ on } \mathbf{SU}(2)$$

for all $x \in \mathbb{H}$ and all $0<t\leq 1$.
For the $\mathbf{SL}(2,\mathbb{R})$ case, we can do the same in small times since we have then a similar Harnack inequality and that the ball-box is a general theorem which for all subelliptic situations. Note also, as we have seen it before, that for the goup $\mathbf{SL}(2,\mathbb{R})$ we have a better ultracontractivity than (5.2.29) valid for all $t>0$ given by inequality (4.6.16).

5.2.3 Some isoperimetrics inequality

We can now recover some isoperimetric results from the Li-Yau inequality. We use methods of Varopoulos and Ledoux (see [91] and [67]). First we set:

Proposition 5.2.5. *For any smooth function f on G,*

$$||\sqrt{\Gamma P_t f}||_\infty \leq \sqrt{3B(t)}||f||_\infty.$$

and

$$||f-P_tf||_1 \leq \int_0^t \sqrt{3B(s)}ds||\sqrt{\Gamma f}||_1.$$

Proof. Indeed, for the first point, the Li-Yau inequality (5.2.23) gives for $0<t<1$ and f a positive function:

$$L(P_tf)^- \leq \frac{B(t)}{A(t)}P_tf.$$

By integrating against μ and noticing $\int L(P_tf)d\mu = 0$, we get

$$\frac{1}{2}\int |L(P_tf)|d\mu \leq \frac{B(t)}{A(t)}\int f d\mu.$$

Then $||LP_tf||_1 \leq \frac{2B(t)}{A(t)}||f||_1$, and since LP_t is auto-adjoint, by duality $||LP_tf||_\infty \leq \frac{2B(t)}{A(t)}||f||_\infty$. By plugging-in this result in the Li-Yau equation (5.2.23), one gets

$$\begin{aligned}
\Gamma(P_tf) &= \Gamma(\ln P_tf)(P_tf)^2 \\
&\leq A(t)||LP_tf||_\infty P_tf + B(t)(P_tf)^2 \\
&\leq \left(A(t)\frac{2B(t)}{A(t)} + B(t)\right)||f||_\infty^2 \\
&\leq 3B(t)||f||_\infty^2
\end{aligned}$$

which implies the first result.
For the second point, let f and g be two smooth functions,

$$\begin{aligned}\int g(P_t f - f) d\mu &= \int_0^t \frac{d}{dt} \int g(P_t f - f) d\mu \\ &= \int_0^t \int g L P_s f d\mu ds \\ &= -\int_0^t \int \Gamma(P_s g, f) d\mu ds\end{aligned}$$

Since $\Gamma(P_s g, f) \leq \sqrt{\Gamma P_s g}\sqrt{\Gamma f}$, by the first point, we have

$$|\int g(P_t f - f) d\mu| \leq ||g||_\infty \int_0^t \sqrt{3B(s)} ds \int \sqrt{\Gamma f} d\mu$$

Letting g tends to $sign(P_t f - f)$ ends the proof. □

And actually these last results will enable us to obtain some isoperimetric inequalities. For A and B measurable sets, let us denote

$$K_t(A, B) = \int_B P_t(1_A) d\mu.$$

It is easy to see that

$$K_t(A, A^c) = \mu(A) - K_t(A, A),$$

$$K_t(B, A) = K_t(A, B)$$

and

$$K_t(A, A) = ||P_{\frac{t}{2}} 1_A||_2^2.$$

We have the following proposition:

Proposition 5.2.6. *Let A be a measurable set of G which is a Caccioppoli set and call $P(A)$ its perimeter (see [47] and the references therein to see their definition in our context) then*

$$K_t(A, A^c) \leq \int_0^t \sqrt{3B(s)} ds P(A). \tag{5.2.30}$$

Remark 5.2.7. *In our setting, if A is a Caccioppoli set, its perimeter is given, in fact, by:*

$$P(A) = \sup_{\phi \in \mathcal{F}} \int_{\mathbb{G}} \mathbf{1}_A \left(\sum_{i=1}^d X_i \phi_i \right) d\mu$$

where

$$\mathcal{F} = \left\{ \phi = (\phi_1, \dots, \phi_d) \in \mathcal{C}_0^1(\mathbb{G}, \mathbb{R})^d, \|\phi\|_\infty = \sup_{\mathbb{G}} \sqrt{\sum_{i=1}^d \phi_i^2} \leq 1 \right\}.$$

Proof. Let A be a measurable set of $\mathbb{G}$ and let f and g be two smooth functions which aproximate respectively 1_A and 1_{A^c} and with $||g||_\infty \leq 1$. Then the quantity $\int g(P_t f - f)d\mu$ converges towards $K_t(A, A^c)$ (note indeed $\inf fgd\mu$ goes to 0) and as before

$$\begin{aligned}\int g(P_t f - f)d\mu &\leq ||g||_\infty ||P_t f - f||_1 \\ &\leq \int_0^t \sqrt{3B(s)}ds \int \sqrt{\Gamma f}d\mu\end{aligned}$$

As it is well known, we can choose f such that $\int \sqrt{\Gamma f}d\mu$ tends towards $P(A)$ (see theorem 1.14 of [47]), so we obtain

$$K_t(A, A^c) \leq \int_0^t \sqrt{3B(s)}dsP(A). \tag{5.2.31}$$

□

Now we are going to specialize the result for each model space. First we begin with the Heisenberg group.

Proposition 5.2.8. *Let A be a measurable set of $\mathbb{H}$ satisfying the above conditions, then*

$$\mu(A)^{\frac{Q-1}{Q}} \leq CP(A)$$

for some positive constant C and $Q = 4$ stands for the homegenous dimension of the group.

Proof. Let A such a measurable set. Since $\int_0^t \sqrt{3B(s)}ds = C\sqrt{t}$ for some constant C, inequality (5.2.31) in this particular case gives:

$$K_t(A, A^c) \leq C\sqrt{t}P(A).$$

Therefore,

$$P(A) \geq \frac{C'}{\sqrt{t}}(\mu(A) - ||P_{\frac{t}{2}} 1_A||_2^2).$$

Using the ultracontractivity in small time, we get $||P_t f||_\infty \leq \frac{C}{t^{Q/2}}||f||_1$ and by interpolation $||P_t f||_2 \leq \frac{\sqrt{C}}{t^{Q/4}}||f||_1$, so

$$P(A) \geq \frac{C'}{\sqrt{t}}\mu(A)\left(1 - \frac{C}{(\frac{t}{2})^{Q/2}}\mu(A)\right).$$

Now we will have to optimise the function of t on the right-hand side. We see this function attains a positive maximum for t of the order $\mu(A)^{\frac{2}{Q}}$ which has value of order $\mu(A)^{\frac{Q-1}{Q}}$. □

Now, let us look at what happens on $\mathbf{SL}(2, \mathbb{R})$. In fact for t small, the constants $A(t)$ and $B(t)$ on $\mathbf{SL}(2, \mathbb{R})$ are of the same order as the ones of $\mathbb{H}$. As a consequence the last proof work also for $\mathbf{SL}(2, \mathbb{R})$. The only difference is, as we restrict ourselves to $0 < t \leq 1$, the final optimisation argument works only for sets A such that $\mu(A)$ is small enough to ensure the positive maxmimum of the function is attained for $t \leq 1$.

Proposition 5.2.9. *Let A be a measurable set of $\mathbf{SL}(2, \mathbb{R})$ satisfying the above conditions and such $\mu(A)$ is small enough, then*

$$\mu(A)^{\frac{Q-1}{Q}} \leq CP(A)$$

for some positive constant C and $Q = 4$ stands for the homegenous dimension of the group.

Remark 5.2.10. *It is known that the result of Proposition 5.2.6 is true for all sets (see theorem 7.5 of [31] and note that* $\mathbf{SL}(2,\mathbb{R})$ *has a constant curvature* $R=-1$*). But with this method, we can not have the right order for sets A such* $\mu(A)$ *is big. It seems that our proposition 5.1.1 is far from being optimal in big times when* $\rho<0$.

In the case $\rho>0$ (the $\mathbf{SU}(2)$ case), the space is compact, the last proposition is rather meaningless but we can obtain the following interesting result. Here fore simplicity reasons, we choose to work with the normalized invariant measure that is we assume $\mu(\mathbb{G})=1$.

Proposition 5.2.11. *There exist a constant* C *such that for all measurable set* A *with smooth boundary,*

$$\mu(A)(1-\mu(A))\leq C\frac{1}{\sqrt{\rho}}P(A).$$

Proof. Indeed, as

$$\int_0^t\sqrt{\rho}\frac{e^{-2\rho s/3\alpha}}{\sqrt{1-e^{-2\rho s/3\alpha}}}ds=\frac{\sqrt{1-e^{-2\rho t/3\alpha}}}{\sqrt{\rho}},$$

inequality (5.2.31) writes here:

$$K_t(A,A^c)\leq C\frac{\sqrt{1-e^{-2\rho t/3\alpha}}}{\sqrt{\rho}}P(A).$$

But now, letting t go to infinity, as p_t converges to 1, then $P_t(1_A)\to\mu(A)$, then recalling $\mu(\mathbf{SU}(2))=1$,

$$K_t(A,A^c)\to\mu(A)(1-\mu(A)),$$

therefore we obtain

$$\mu(A)(1-\mu(A))\leq C\frac{1}{\sqrt{\rho}}P(A).$$

□

By using the co-area formula and arguments of Buser [27], one has the following L^1-Poincaré inequality:

Proposition 5.2.12. *Let* f *a smooth fuction with* $\int f d\mu=0$*, then*

$$\int|f|d\mu\leq C'\frac{1}{\sqrt{\rho}}\int\sqrt{\Gamma(f)}d\mu.$$

Proof. Let m be a median for f, that is

$$\mu(f\geq m)\geq\frac{1}{2}\text{ and }\mu(f\leq m)\geq\frac{1}{2}.$$

Set

$$f^+=(f-m)_+\text{ and }f^-=(m-f)_+,$$

then

$$f-m=f^+-f^-.$$

We have

$$\int_{\mathcal{M}}|f-m|d\mu=\int_{\mathcal{M}}f^+d\mu+\int_{\mathcal{M}}f^-d\mu,$$

thus, by the co-area formula,

$$\int_{\mathcal{M}} |f-m|d\mu = \int_{\mathcal{M}} \mu(f^+ \geq t)dt + \int_{\mathcal{M}} \mu(f^- \geq t)dt.$$

Observe that, by the median property, for every $t>0$,

$$\mu(f^+ \geq t) \leq \frac{1}{2} \text{ and } \mu(f^- \geq t) \leq \frac{1}{2}.$$

Thus,

$$\frac{1}{1-\mu(f^+ \geq t)} \leq 2 \text{ and } \frac{1}{1-\mu(f^- \geq t)} \leq 2$$

and by Proposition 5.2.11,

$$\mu(f^+ \geq t) \leq \frac{2C}{\sqrt{\rho}} P(f^+ \geq t)$$

and

$$\mu(f^+ \geq t) \leq \frac{2C}{\sqrt{\rho}} P(f^+ \geq t).$$

This gives

$$\int_{\mathcal{M}} |f-m|d\mu \leq \frac{2C}{\sqrt{\rho}} (\int_{\mathcal{M}} \sqrt{\Gamma(f^+)}d\mu + \int_{\mathcal{M}} \sqrt{\Gamma(f^-)}d\mu.$$

Now, as f^+ and f^- have disjoint supports,

$$\sqrt{\Gamma(f^+)} + \sqrt{\Gamma(f^-)} = \sqrt{\Gamma(f^+ + f^-)} = \sqrt{\Gamma(f)}.$$

To conclude and to obtain the desired inequality, we use the following well-known inequality,

$$\frac{1}{2}\int |f - \int f d\mu| d\mu \leq \int |f-m|d\mu = \inf_{a\in\mathbb{R}} \int |f-a|d\mu.$$

$\square$

5.2.4 A gradient bound for the heat kernel

As another corollary of the Harnack inequalities, we can also prove the following global estimate:

Proposition 5.2.13. *Let $\mathbb{G} = \mathbb{H}$ or $\mathbf{SU}(2)$. There exists a constant $C>0$ such that for $t\in(0,1)$, $(r,z)\in\mathbb{G}$,*

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq C\left(\frac{d(r,z)}{t} + \frac{1}{\sqrt{t}}\right),$$

where $d(r,z)$ denotes the Carnot Carathéodory distance from 0 to the point with cylindric coordinates (r,θ,z).

Proposition 5.2.14. *On $\mathbf{SL}(2,\mathbb{R})$, there exists a constant $C'>0$ such that for $t\in(0,1)$, $r>0$, $z\in[-\pi,\pi]$,*

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq C'\left(\frac{d(r,z)}{t} + \frac{1}{\sqrt{t}}\right),$$

and there exists a constant $C''>0$ such that for $t>2$, $r>0$, $z\in[-\pi,\pi]$,

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq C''\left(\frac{d(r,z)}{t} + 1\right),$$

Proof. The proof is based on the Harnack inequalities and on the fact that on each of our favorite spaces, the Γ_2 of a radial function is non negative.
In what follows, we fix $t > 0$ and $(r,z) \in \mathbb{G}$ where $\mathbb{G} = \mathbb{H}$, $\mathbf{SU}(2)$ or $\mathbf{SL}(2,\mathbb{R})$. Let

$$\phi(s) = P_s(p_{t-s} \ln p_{t-s})$$

so that

$$\phi'(s) = P_s(p_{t-s}\Gamma(\ln p_{t-s}))$$

and

$$\phi''(s) = 2P_s(p_{t-s}\Gamma_2(\ln p_{t-s})).$$

Since p_t only depends on (r,z) we have $\Gamma_2(\ln p_{t-s}) \geq 0$ and therefore $\phi''(s) \geq 0$, that is ϕ is convex. By convexity of ϕ, the slopes $\phi'(s)$ are increasing and thus, we obtain:

$$\phi\left(\frac{t}{2}\right) - \phi(0) = \int_0^{\frac{t}{2}} \phi'(s)ds \geq \frac{t}{2}\phi'(0)$$

that is, at the point $g = (r,z)$:

$$p_t(g)\Gamma(\ln p_t)(g) \leq \frac{2}{t}\left(P_{t/2}\left(p_{t/2}\ln p_{t/2}\right)(g) - p_t(g)\ln p_t(g)\right).$$

Now using the facts that $p_{\frac{t}{2}}(r',z') \leq p_{\frac{t}{2}}(0,0)$ for all $(r',z') \in G$ and that

$$\begin{aligned} P_{\frac{t}{2}}\left(p_{\frac{t}{2}}\right)(g) &= \int p_{\frac{t}{2}}(g,g')p_{\frac{t}{2}}(0,g')d\mu(g') \\ &= \int p_{\frac{t}{2}}(0,g')p_{\frac{t}{2}}(g',g)d\mu(g') \\ &= p_t(g), \end{aligned}$$

one obtains the following bound:

$$p_t(g)\Gamma(\ln p_t)(g) \leq \frac{2}{t}\left(p_t(g)\ln p_{\frac{t}{2}}(0) - p_t(g)\ln p_t(g)\right)$$

and therefore

$$\Gamma(\ln p_t)(g) \leq \frac{2}{t}\ln\frac{p_{\frac{t}{2}}(0)}{p_t(g)}.$$

Then one concludes using the different Harnack inequalities of Propositions 5.2.1, 5.2.2 and 5.2.4. □

5.3 A different analysis of the differential inequality and an elliptic gradient

In this section we continue the study of the differential inequality (5.1.9) but in a different way. This time, we choose some functions a and b such that the right hand side of this inequality is positive. We recall here the inequality, with the previous notations, it reads:

$$(a\Phi_1 + b\Phi_2)' \geq \left(a' + (8\rho - \frac{4}{\lambda} - 2\gamma)a\right)\Phi_1 + (4a+b')\Phi_2 + (-2a\lambda + b)\Phi_2' + 2a\gamma LP_tf - a\gamma^2 P_t f$$

where a, b are two general functions, $\lambda > 0$ and $\gamma \in \mathbb{R}$. Hence, choosing $\gamma = 0$, $a(s) = e^{-ks}$, $b(s) = ce^{-ks}$ and $\lambda = \frac{c}{2}$, where c needs to be positive, we obtain:

$$(a\Phi_1 + b\Phi_2)' \geq \left(8(\rho - \frac{1}{c}) - k\right) e^{-ks}\Phi_1 + (4 - ck)e^{-ks}\Phi_2$$

and therefore

$$(a\Phi_1 + b\Phi_2)' \geq 0$$

if

$$k \leq \min\left(8(\rho - \frac{1}{c}), \frac{4}{c}\right).$$

When, $\rho > 0$ the maximum value of this minimum is obtained when the two quantities are equal, thus is obtained when $c = \frac{3}{2\rho}$ and takes value $\frac{8\rho}{3}$. As a consequence when $\rho > 0$, we get the following proposition:

Proposition 5.3.1. *Let L be a diffusion which satisfies the hypothesis of Proposition 5.1.1 with $\rho > 0$. Let f a smooth function on $\mathcal{M}$, then*

$$P_t(f)\left(\Gamma(\ln P_t f) + \frac{3}{2\rho}Z(\ln P_t f)^2\right) \leq e^{-\frac{8\rho t}{3}} P_t\left(f\left(\Gamma(\ln f) + \frac{3}{2\rho}Z(\ln f)^2\right)\right).$$

When $\rho \leq 0$, there is no way to optimize in c since the better value is obtained when c goes to infinity. But, one can take $c = 1$ and $k = 8(\rho - 1)$ and get:

Proposition 5.3.2. *Let L be a diffusion which satisfies the hypothesis of Proposition 5.1.1 with $\rho \leq 0$. Let f a smooth function on $\mathcal{M}$, then*

$$P_t(f)\left(\Gamma(\ln P_t f) + Z(\ln P_t f)^2\right) \leq e^{|8(\rho-1)|t} P_t\left(f\left(\Gamma(\ln f) + Z(\ln f)^2\right)\right).$$

Note also that as we only use $\gamma = 0$, we do not take in account the dimensional term $\frac{1}{2}(Lf)^2$. Therefore, we can consider the functions

$$\Psi_1(s) = P_s\left(\Gamma(P_{t-s}f)\right),$$

$$\Psi_2(s) = P_s\left(Z(P_{t-s}f)^2\right)$$

whose derivatives are given by

$$\Psi_1'(s) = 2P_s\left(\Gamma_2(P_{t-s}f)\right),$$

$$\Psi_2'(s) = 2P_s\left(\Gamma(Z(P_{t-s}f))\right).$$

More generally, the derivative of the function $\Psi_V(s) = P_s\left(V(P_{t-s}f)^2\right)$ for a general vector field V is

$$\Psi_V'(s) = 2P_s\big(\Gamma(V(P_{t-s}f)) + V(P_{t-s}f)\,[L, V](P_{t-s}f)\big).$$

We then get the differential inequality:

$$(a\Psi_1 + b\Psi_2)' \geq \left(a' + (8\rho - \frac{4}{\lambda})a\right)\Psi_1 + (4a + b')\Psi_2 + (-2a\lambda + b)\Psi_2'.$$

By the same argument, we then obtain:

Proposition 5.3.3. *Let L be a diffusion operator which satisfies the hypothesis of Proposition 5.1.1 with $\rho > 0$. Let f a smooth function on $\mathcal{M}$, then*

$$\Gamma(P_t f) + \frac{3}{2\rho} Z(P_t f)^2 \leq e^{-\frac{8\rho t}{3}} P_t \left(\Gamma(f) + \frac{3}{2\rho} Z(f)^2 \right).$$

Proposition 5.3.4. *Let L be a diffusion operator which satisfies the hypothesis of Proposition 5.1.1 with $\rho \leq 0$. Let f a smooth function on $\mathcal{M}$, then*

$$\Gamma(P_t f) + Z(P_t f)^2 \leq e^{|8(\rho-1)|t} P_t \left(\Gamma(f) + Z(f)^2 \right).$$

Now we describe a different way to get these last results. Indeed, consider a diffusion operator L which satisfies the hypothesis of Proposition (5.1.1) and let us denote, for a real number $c \geq 0$

$$\tilde{\Gamma}(f) = \Gamma(f) + cZ(f)^2.$$

We can define:

$$\tilde{\Gamma_2}(f) = \frac{1}{2} \left(L\tilde{\Gamma}(f,f) - 2\tilde{\Gamma}(f, Lf) \right).$$

Then we have the following proposition:

Proposition 5.3.5. *Let $k \in \mathbb{R}$, then the following proposition are equivalent:*

- $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$, $\tilde{\Gamma_2}(f) \geq k\tilde{\Gamma}(f)$
- $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$, $\tilde{\Gamma}(P_t f) \leq e^{-2kt} P_t(\tilde{\Gamma}(f))$
- $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$, $(P_t f)\tilde{\Gamma}(\ln P_t f) \leq e^{-2kt} P_t(f\tilde{\Gamma}(\ln f))$.

Proof. The proof is the same than the classical one of Theorem 7.1.1 once one notices that under our hypothesis,

$$\frac{d}{ds} P_s(\tilde{\Gamma}(P_{t-s} f)) = 2P_s(\tilde{\Gamma_2}(P_{t-s} f))$$

and

$$\frac{d}{ds} P_s(P_{t-s} f \tilde{\Gamma}(\ln P_{t-s} f)) = 2P_s(P_{t-s} f \tilde{\Gamma_2}(\ln P_{t-s} f)).$$

Then one get the direct sense using Gronwall lemma. For the reverse sense, the inequalities are in fact equalities when $t = 0$ so that one can compare their derivatives in $t = 0$. □

Remark 5.3.6. *In fact this last proposition is true if only the antisymmetric condition (5.1.2) is satisfied.*

We can now obtain some Poincaré and log-Sobolev inequalities under this criterion but involving the operator $\tilde{\Gamma}$.

Proposition 5.3.7. *Under the previous hypothesis, if $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$, $\tilde{\Gamma_2}(f) \geq k\tilde{\Gamma}(f)$, then*

$$\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R}), P_t(f^2) - P_t(f)^2 \leq \frac{1 - e^{-2kt}}{k} P_t(\tilde{\Gamma}(f))$$

and

$$P_t(f \log(f)) - P_t(f) \log(P_t(f)) \leq \frac{1 - e^{-2kt}}{k} P_t \left(\frac{\tilde{\Gamma}(f)}{f} \right).$$

Proof. We do the proof only for the Poincaré inequality, it is similar for the log-Sobolev inequality. Let $f \in \mathcal{C}_c^\infty(\mathcal{M}, \mathbb{R})$,

$$\begin{aligned} P_t(f^2) - P_t(f)^2 &= \int_0^t \frac{d}{ds} P_s((P_{t-s}f)^2) ds \\ &= 2\int_0^t P_s(\Gamma(P_{t-s}f)) ds \\ &\leq 2\int_0^t P_s(\tilde{\Gamma}(P_{t-s}f)) ds \\ &\leq 2\int_0^t e^{-k(t-s)} ds P_t(\tilde{\Gamma}(f)) \\ &\leq \frac{1-e^{-2kt}}{k} P_t(\tilde{\Gamma}(f)). \end{aligned}$$

□

Remark 5.3.8. *In our setting, for all $\lambda > 0$,*

$$\tilde{\Gamma_2} \geq \frac{1}{2}(Lg)^2 + 2(Zg)^2 + \left(4\rho - \frac{2}{\lambda}\right)\Gamma(g) + (c - 2\lambda)\Gamma(Zg)$$

and then with $\lambda = \frac{c}{2}$,

$$\tilde{\Gamma_2} \geq 2(Zg)^2 + \left(4(\rho - \frac{1}{c})\right)\Gamma(g).$$

The goal now is to find the best k such that $\tilde{\Gamma_2} \geq k\tilde{\Gamma}$. This means first find the best $c > 0$ such that

$$\min\left(\frac{2}{c}, 4(\rho - \frac{1}{c})\right)$$

is maximum. Then the value of this maximum gives the desired k. This is the same problem that we just treat and we find also $k = \frac{4\rho}{3}$ when $\rho > 0$ and we can choose $c = 1$ and $k = 4(\rho - 1)$ when $\rho \leq 0$. Note that one can also consider $\tilde{\Gamma}(f) = \Gamma(f) + cZ(f)^2$ with $c < 0$ when $\rho < 0$. One then gets

$$\tilde{\Gamma_2} \geq k\tilde{\Gamma}$$

with $k = \frac{4\rho}{3} < 0$. In this case we can not obtain poincaré and log-Sobolev inequality but some strange reverse and log-Sobolev inequality.

Chapter 6

The reverse Poincaré inequality

In this chapter we will prove two functional inequalities which are concerned about regularisation properties of the semi-group. In the proofs of these inequalities we rely heavily on the fact that our three model spaces are Lie groups with a left-invariant metric. Moreover, the symmetries described in Section 2.6 enable us to get the optimal constants in these inequalities.

6.1 A first gradient bound

In all the chapter, the letter $\mathbb{G}$ will denote one of the groups $\mathbb{H}, \mathbf{SU}(2)$ or $\mathbf{SL}(2, \mathbb{R})$.

Proposition 6.1.1. *Let $f : \mathbb{G} \to \mathbb{R}$ be a smooth function. For $t > 0$ and $g \in \mathbb{G}$,*

$$\Gamma(P_t f, P_t f)(g) \leq A(t) \left(\int_{\mathbb{G}} f^2 d\mu - \left(\int_{\mathbb{G}} f d\mu \right)^2 \right) \tag{6.1.1}$$

where

$$A(t) = -\frac{1}{4} \frac{d}{dt} \int_{\mathbb{G}} p_t^2 d\mu.$$

Proof. By left invariance, it is enough to prove this inequality at $g = 0$. We can moreover assume that $\int_{\mathbb{G}} f d\mu = 0$. Let us denote by $\hat{X}$ and $\hat{Y}$ the right invariant vector fields, they commute with P_t. As the left invariant and the right vector fields coincide at the identity, we have:

$$X P_t(f)(0) = \hat{X} P_t(f)(0) = P_t(\hat{X} f)(0).$$

Now, we may write

$$\begin{aligned} \Gamma(P_t f, P_t f)(0) &= \sup_{a^2+b^2=1} \left(a X P_t(f)(0) + b Y P_t(f)(0)\right)^2 \\ &= \sup_{a^2+b^2=1} \left(a P_t(\hat{X} f)(0) + b P_t(\hat{Y} f)(0)\right)^2. \end{aligned}$$

But, for a and b such that $a^2 + b^2 = 1$,

$$\begin{aligned} a P_t(\hat{X} f)(0) + b P_t(\hat{Y} f)(0) &= \int_{\mathbb{G}} (a\hat{X} f + b\hat{Y} f)(r, \theta, z) p_t(r, z) d\mu(r, \theta, z) \\ &= -\int_{\mathbb{G}} (a\hat{X} p_t + b\hat{Y} p_t)(r, z) f(r, \theta, z) d\mu(r, \theta, z). \end{aligned}$$

Therefore, by Cauchy-Schwarz inequality,

$$\left(aP_t(\hat{X}f)(0) + bP_t(\hat{Y}f)(0)\right)^2 \leq \int_{\mathbb{G}} (a\hat{X}p_t + b\hat{Y}p_t)^2 d\mu \int_{\mathbb{G}} f^2 d\mu$$

Now, since p_t does not depend on θ, by the symmetrical considerations of Section 2.6, one has

$$\int_{\mathbb{G}} (a\hat{X}p_t + b\hat{Y}p_t)^2 d\mu = \int_{\mathbb{G}} (\hat{X}p_t)^2 d\mu = \int_{\mathbb{G}} (\hat{Y}p_t)^2 d\mu = \frac{1}{2}\int_{\mathbb{G}} \hat{\Gamma}(p_t, p_t) d\mu = \frac{1}{2}\int_{\mathbb{G}} \Gamma(p_t, p_t) d\mu$$

Thus one can conclude

$$\Gamma(P_t f, P_t f)(0) \leq \frac{1}{2}\int_{\mathbb{G}} \Gamma(p_t, p_t) d\mu \int_{\mathbb{G}} f^2 d\mu,$$

which is the required inequality because:

$$\int_{\mathbb{G}} \Gamma(p_t, p_t) d\mu = -\int_{\mathbb{G}} p_t L p_t d\mu = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{G}} p_t^2 d\mu.$$

□

Remark 6.1.2. *We can note using the semigroup property and and the relation* $p_t(0, g) = p_t(g, 0)$ *that we also have:*

$$\int_{\mathbb{G}} p_t^2 d\mu = p_{2t}(0).$$

Remark 6.1.3. *Equality is achieved in (6.1.1) when* $f = \hat{X}(p_t)$ *for instance or more generally for all functions* $f = a\hat{X}p_t + b\hat{Y}p_t$ *since we have an equality for them in the Cauchy-Schwarz inequality.*

We now study the constant $A(t)$. First we show the following property.

Proposition 6.1.4. *On the three spaces* $\mathbb{H}, \mathbf{SU}(2)$ *and* $\mathbf{SL}(2, \mathbb{R})$ *the constant* $A(t)$ *is decreasing.*

Proof. Let us compute the derivative of A.

$$\begin{aligned} A'(t) &= \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{G}} \Gamma(p_t, p_t) d\mu \\ &= \frac{\partial}{\partial t}\int_{\mathbb{G}} \Gamma(Lp_t, p_t) d\mu \\ &= -\int_{\mathbb{G}} \Gamma_2(p_t, p_t) d\mu. \end{aligned}$$

since by definition of Γ_2, $\Gamma_2(p_t, p_t) = \frac{1}{2}L\Gamma(p_t, p_t) - \Gamma(Lp_t, p_t)$ and since $\int_{\mathbb{G}} Lf d\mu = 0$ for all smooth functions f.

Now since p_t only depends on (r, z), $\Gamma_2(p_t, p_t) \geq 0$ and thus $A'(t) \leq 0$. □

The computation of the constant $A(t)$ can be done by an explicit computation on the Heisenberg group $\mathbb{H}$.

Proposition 6.1.5. *On the Heisenberg group,*

$$A(t) = \frac{1}{256t^3} \text{ for all } t > 0.$$

Proof. Indeed

$$A(t) = -\frac{1}{4}\frac{d}{dt}\int_{\mathbb{G}} p_t^2 d\mu = -\frac{1}{4}\frac{d}{dt}p_{2t}(0)$$

and

$$h_t(0) = \frac{1}{32t^2}.$$

□

Now let us look the behaviour of the constant $A(t)$ in small and big times on the two other model spaces. First we look at the small times, we have

Proposition 6.1.6. *When t goes to 0, on $\mathbb{H}, \mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, one has*

$$A(t) \sim_{t\to 0} \frac{1}{256t^3}. \tag{6.1.2}$$

Remark 6.1.7. *Actually as we have seen it, there is an equality on the Heisenberg group in (6.1.2) and it is valid for all $t > 0$.*

On the two other spaces $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, to obtain the behaviour of $A(t)$ in small times, we can proceed both way. We can do an explicit computation (that is the method we will use here) or we can use the convergence of the diffusion towards the one on the Heisenberg group (we will use this method to express the behaviour of the constant in the reverse Poincaré in small times, see the proof of Proposition 6.2.7).

Proof. Recall on the three models,

$$A(t) = -\frac{1}{4}\frac{d}{dt}\int_{\mathbb{G}} p_t^2 d\mu = -\frac{1}{4}\frac{d}{dt}p_{2t}(0,0).$$

Now on $\mathbf{SU}(2)$, one has

$$\begin{aligned} p_t(0,0) &= \frac{1}{2\pi^2}\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}(2k+\mid n\mid+1)e^{-(4k(k+|n|+1)+2|n|)t} \\ &= \frac{e^t}{8t^2}\sum_{k\in\mathbb{Z}} e^{-\frac{k(k+1)\pi^2}{t}}\frac{(2k+1)+2ke^{-\frac{2k\pi^2}{t}}}{\left(1+e^{-\frac{k\pi^2}{t}}\right)^2}, \end{aligned}$$

and on $\mathbf{SL}(2,\mathbb{R})$, one has

$$p_t(0,0) = \frac{e^{-t}}{32t^2}.$$

The result follows then by easy computations. □

These last eplicit expressions of $p_t(0,0)$ enable us also to obtain the behaviour in big times.

Proposition 6.1.8. *When t goes to ∞, we have the following behaviours:*

- *On $\mathbb{H}$, $A(t) = \frac{1}{256t^3}$, for all $t > 0$.*
- *On $\mathbf{SU}(2)$, $A(t) \sim_{t\to+\infty} \frac{e^{-4t}}{\pi^2}$.*
- *On $\mathbf{SL}(2,\mathbb{R})$, $A(t) \sim_{t\to+\infty} \frac{e^{-2t}}{256t^2}$.*

6.2 The reverse Poincaré inequality

Now by doing a similar study, we can obtain a reverse Poincare inequality with a sharp constant for the subelliptic heat kernel measure on the three model spaces.

Proposition 6.2.1. *Let $f : \mathbb{G} \to \mathbb{R}$ be a smooth function. For $t > 0$ and $g \in \mathbb{G}$,*

$$\Gamma(P_t f, P_t f)(g) \leq C(t) \left(P_t f^2(g) - (P_t f)^2(g) \right) \tag{6.2.3}$$

where

$$C(t) = -\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{G}} p_t \ln p_t d\mu.$$

Proof. The proof is very close from the one of proposition 6.1.1 The only difference is that we will use the Cauchy-Schwarz inequality against the measure $p_t d\mu$ instead of against the measure $d\mu$. Let us do th proof. As before it is enough to prove this inequality at $g = 0$ and we can assume that $\int_{\mathbb{G}} f d\mu = 0$. Now, as before, we may write

$$\begin{aligned} \Gamma(P_t f, P_t f)(0) &= \sup_{a^2+b^2=1} \left(aXP_t(f)(0) + bYP_t(f)(0) \right)^2 \\ &= \sup_{a^2+b^2=1} \left(aP_t(\hat{X}f)(0) + bP_t(\hat{Y}f)(0) \right)^2. \end{aligned}$$

and for a and b such that $a^2 + b^2 = 1$,

$$aP_t(\hat{X}f)(0) + bP_t(\hat{Y}f)(0) = -\int_{\mathbb{G}} (a\hat{X}p_t + b\hat{Y}p_t)(r, z) f(r, \theta, z) d\mu(r, \theta, z).$$

Therefore, by Cauchy-Schwarz inequality against the measure $p_t d\mu$, we have

$$\left(aP_t(\hat{X}f)(0) + bP_t(\hat{Y}f)(0) \right)^2 \leq \int_{\mathbb{G}} (a\hat{X}p_t + b\hat{Y}p_t)^2 \frac{1}{p_t} d\mu \int_{\mathbb{G}} f^2 p_t d\mu$$

But

$$\int_{\mathbb{G}} (a\hat{X}p_t + b\hat{Y}p_t)^2 \frac{1}{p_t} d\mu = \int_{\mathbb{G}} (a\hat{X} \ln p_t + b\hat{Y} \ln p_t)^2 p_t d\mu$$

and

$$\int_{\mathbb{G}} f^2 p_t d\mu = P_t(f^2)(0).$$

Now, as before since p_t does not depend on θ, by the symmetric considerations of Section 2.6, one has

$$\int_{\mathbb{G}} (a\hat{X} \ln p_t + b\hat{Y} \ln p_t)^2 p_t d\mu = \frac{1}{2} \int_{\mathbb{G}} \hat{\Gamma}(\ln p_t, \ln p_t) p_t d\mu = \frac{1}{2} \int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu.$$

Thus one can conclude

$$\Gamma(P_t f, P_t f)(0) \leq \frac{1}{2} \int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu \, P_t(f^2),$$

and one can note the following equalities

$$\int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu = \int_{\mathbb{G}} \Gamma(\ln p_t, p_t) d\mu = -\int_{\mathbb{G}} \ln p_t L p_t d\mu = -\frac{\partial}{\partial t} \int_{\mathbb{G}} p_t \ln p_t d\mu.$$

□

Remark 6.2.2. *Equality is achieved in (6.2.3) when $f = \hat{X}(\ln p_t)$ for instance or more generally for all functions $f = a\hat{X}\ln p_t + b\hat{Y}\ln p_t$ since we have an equality for them in the Cauchy-Schwarz inequality.*

We now study the constant

$$C(t) = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{G}} p_t \ln p_t d\mu.$$

Proposition 6.2.3. *On the three spaces $\mathbb{H}, \mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ the constant $C(t)$ is decreasing.*

Proof. To show that C is decreasing, let us compute its derivative, we have

$$\begin{aligned}
2C'(t) &= \frac{d}{dt}\int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu \\
&= 2\int_{\mathbb{G}} \Gamma(\frac{Lp_t}{p_t}, \ln p_t) p_t d\mu + \int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) Lp_t d\mu \\
&= 2\int_{\mathbb{G}} \Gamma(L\ln p_t, \ln p_t) p_t d\mu + 2\int_{\mathbb{G}} \Gamma\left(\Gamma(\ln p_t, \ln p_t), p_t\right) d\mu + \int_{\mathbb{G}} L\Gamma(\ln p_t, \ln p_t) p_t d\mu \\
&= 2\int_{\mathbb{G}} \Gamma(L\ln p_t, \ln p_t) p_t d\mu - 2\int_{\mathbb{G}} L\Gamma(\ln p_t, \ln p_t) p_t d\mu + \int_{\mathbb{G}} L\Gamma(\ln p_t, \ln p_t) p_t d\mu \\
&= -\int_{\mathbb{G}} \Gamma_2(\ln p_t, \ln p_t) p_t d\mu.
\end{aligned}$$

But now, let us observe that p_t only depends on (r, z). Therefore $\Gamma_2(\ln p_t, \ln p_t) \geq 0$ and thus $C'(t) \leq 0$.

□

As before, we can do an explicit computation for the constant $C(t)$. This computation is a little less immediate than the one for the constant $A(t)$ and use the dilation structure of $\mathbb{H}$.

Proposition 6.2.4. *On the Heisenberg group,*

$$C(t) = \frac{1}{t} \text{ for all } t > 0.$$

Proof. To compute this constant, we use the dilation operator D and the formula (2.4.15). Let us recall it:

$$P_t D = DP_t + tP_t L.$$

As the dilation vector field D vanishes in 0, for all $t > 0$ and for any smooth f,

$$P_t((tL - D)f)(0) = 0,$$

that is

$$\int (tL - D)f \; h_t \, dx = 0.$$

This means

$$(tL + D + 2)h_t = 0$$

since the adjoint of D is $-D - 2$.

Multiply then by $\ln h_t$ and taking integral gives

$$\int \ln h_t \; (tL + D + 2)h_t \, dx = 0.$$

But by using an integration by parts,

$$\int \ln h_t \; tLh_t \, dx = -t \int \Gamma(\ln h_t, \ln h_t) \; h_t \, dx.$$

Moreover, we have

$$\int \ln h_t \; (D+2)h_t \, dx = -\int h_t \; D \ln h_t \, dx = -\int Dh_t dx = 2 \int h_t \, dx = 2$$

and therefore

$$\int \Gamma(\ln h_t, \ln h_t) \; h_t \, dx = \frac{2}{t}.$$

□

Remark 6.2.5. *Note the difference with the elliptic case: for the heat semigroup $(P_t)_{t \geq 0}$ in $\mathbb{R}^n$ or for any manifold with non negative Ricci curvature, one has for every $t \geq 0$ and any smooth f,*

$$2t\Gamma(P_t f, P_t f) \leq P_t(f^2) - (P_t f)^2.$$

Remark 6.2.6. *This proof of the reverse Poincaré inequality with optimal constant can be generalized on some nilpotent left-invariant Lie groups admitting dilations. Those groups are called Carnot groups.*
Indeed if a Lie group $\mathbb{G}$ admits a sublaplacian which can be written $L = \sum_{i=1}^{n_0} X_i^2$ and which satisfies the rotational invariance

$$\int_{\mathbb{G}} (\sum_{i=1}^{n_0} a_i \hat{X}_i \ln p_t)^2 p_t d\mu = \frac{1}{n_0} \int_{\mathbb{G}} \hat{\Gamma}(\ln p_t, \ln p_t) p_t d\mu = \frac{1}{n_0} \int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu$$

for all a_i such that $\sum_{i=1}^{n_0} a_i^2 = 1$ and if moreover there exists a dilation vector field D on $\mathbb{G}$ (that is a vector field D such that $[L, D] = L$) whose adjoint is

$$D^* = -D - \frac{Q}{2} Id;$$

then the preceding proof still works and gives the optimal constant $\frac{Q}{2n_0}$. Therefore in this setting, one has the sharp inequality:

$$\Gamma(P_t f, P_t f) \leq \frac{Q}{2n_0} (P_t(f^2) - (P_t f)^2). \tag{6.2.4}$$

The quantity Q is, in fact, the homogenous dimension of the group G.
This last inequality (6.2.4) recovers at the same time the Heisenberg and Euclidean cases.
For the $(2p+1)$-dimensional Heisenberg group $\mathbb{H}_{2p+1}$, the two conditions are clearly sastisfied and we have $n_0 = 2p$ while the homogeneous dimension is $Q = 2p+2$. Therefore here the inequality writes

$$\Gamma(P_t f, P_t f) \leq \frac{p+1}{2pt} (P_t(f^2) - (P_t f)^2)$$

and the constant approaches the Euclidean one when p goes to infinity.

Now let us look at the behaviour of the constant $C(t)$ in small and big times on the two other model spaces. First we look at the small times, we have

Proposition 6.2.7. *When t goes to 0, on $\mathbb{H}, \mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, one has*

$$C(t) \sim_{t\to 0} \frac{1}{t}. \tag{6.2.5}$$

Remark 6.2.8. *Actually as we have seen it, there is an equality on the Heisenberg group in (6.2.5) and it is valid for all $t>0$.*

This time, to obtain the behaviour of the constant $C(t)$ in small times on $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, it seems difficult to use a direct computation. Rather we should use the convergence of the diffusion on those spaces towards the one on the Heisenberg group.

Proof. We do the proof for both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ in the same time. As usual we denote by p_t the heat kernel on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ and by h_t the heat kernel on the Heisenberg group. We also denote by Γ on the associated operator on on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ and $\Gamma^{\mathbb{H}}$ the one on $\mathbb{H}$. The idea of the proof is that, asymptotically when $t\to 0$, the constant $C(t)$ has to behave like the best constant of the reverse spectral gap inequality on the Heisenberg group which we just compute before. Let $0<t<1$ we have:

$$\begin{aligned} tC(t) &= \frac{t}{2}\int_{\mathbb{G}} p_t\Gamma(\ln p_t, \ln p_t)d\mu \\ &= \int_{A_{r,z}} t^{5/2}s(t,r)p_t(\sqrt{t}r, tz)\Gamma(\ln p_t, \ln p_t)(\sqrt{t}r,tz)drdz \end{aligned}$$

where $A_{r,z}$ equals $[0,\frac{\pi}{2\sqrt{t}}]\times[-\frac{\pi}{t},\frac{\pi}{t}]$ on $\mathbf{SU}(2)$ and $[0,\infty[\times[-\frac{\pi}{t},\frac{\pi}{t}]$ on $\mathbf{SL}(2,\mathbb{R})$ and $s(t,r)$ equals $\frac{\sin 2\sqrt{t}r}{2}$ on $\mathbf{SU}(2)$ and $\frac{\sinh 2\sqrt{t}r}{2}$ on $\mathbf{SL}(2,\mathbb{R})$. Now, by using the result of Section 4.7, we easily obtain that for both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, uniformly on compact sets, the following convergences hold

$$\lim_{t\to 0} t^{3/2}s(t,r)p_t(\sqrt{t}r,tz) = h_1(r,z)r$$

and

$$\lim_{t\to 0} t\Gamma(\ln p_t, \ln p_t)(\sqrt{t}r,tz) = \Gamma^{\mathbb{H}}(\ln h_1)(r,z).$$

So we obtain the desired convergence on any compact subsets $[0,R]\times[-A,A]$, that is

$$\begin{aligned} &\int_0^R\int_{-A}^A t^{5/2}\frac{\sinh 2\sqrt{t}r}{2}p_t(\sqrt{t}r,tz)\Gamma(\ln p_t,\ln p_t)(\sqrt{t}r,tz)drdz \\ \to_{t\to 0} &\int_0^R\int_{-A}^A h_1(r,z)\Gamma^{\mathbb{H}}(\ln h_1)(r,z)rdrdz. \end{aligned}$$

Now we have also to control the integrand on the outside of the compact K. Thanks to Proposition 5.2.13, on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ there exists a constant $C>0$ such that

$$t\Gamma(\ln p_t,\ln p_t)(\sqrt{t}r,tz) \le C\left(1+\frac{d(\sqrt{t}r,tz)}{\sqrt{t}}\right)^2, \quad t\in(0,1).$$

And thanks to proposition 4.6.30, on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ there exists two constant $C_1, C_2>0$ such that

$$t^2p_t(\sqrt{t}r,tz) \le C_1\exp\left(-C_2\frac{d^2(\sqrt{t}r,tz)}{t}\right).$$

Also we have

$$s(t,r)=\frac{\sin 2\sqrt{t}r}{\sqrt{t}}\leq 2r \text{ on } \mathbf{SU}(2)$$

and

$$s(t,r)=\frac{\sinh 2\sqrt{t}r}{\sqrt{t}}\leq e^{2r} \text{ on } \mathbf{SL}(2,\mathbb{R}).$$

Eventually, the estimates of the distance of Propositions 4.6.27 and 4.6.29 show that on both spaces $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, the integral outside the compact tends to 0 when t goes to 0. Therefore, on both spaces $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$:

$$\lim_{t\to 0} tC(t)=\frac{1}{2}\int_{\mathbb{R}^3} h_1(r,z)\Gamma^{\mathbb{H}}(\ln h_1)(r,z)rdrd\theta dz.$$

Evantually, the value of this last quantity has been calculated in proposition 6.2.4 and is $\frac{1}{t}$. □

Proposition 6.2.9. *When t goes to ∞, we have the following behaviours::*

- *On $\mathbb{H}$, $C(t)=\frac{1}{t}$ for all $t>0$.*
- *On, $\mathbf{SU}(2)$, $C(t)\sim_{t\to+\infty} 4e^{-4t}$.*

Proof. For that, we use the expression

$$C(t)=\frac{1}{2}\int_{\mathbf{SU}(2)}\frac{\Gamma(p_t,p_t)}{p_t}d\mu$$

and the spectral decomposition of Proposition 4.2.1 to get that uniformly on $\mathbf{SU}(2)$,

$$\Gamma(p_t,p_t)\sim_{t\to+\infty}\frac{16e^{-4t}}{(2\pi^2)^2}\Gamma(\cos r\cos z,\cos r\cos z)$$

Therefore, since $p_t\to\frac{1}{2\pi^2}$ uniformly,

$$C(t)\sim_{t\to\infty} 8e^{-4t}\int_{\mathbf{SU}(2)}\Gamma(\cos r\cos z,\cos r\cos z)\frac{d\mu}{2\pi^2},$$

and we compute

$$\int_{\mathbf{SU}(2)}\Gamma(\cos r\cos z,\cos r\cos z)\frac{d\mu}{2\pi^2}=\frac{1}{2},$$

to conclude. □

Remark 6.2.10. *We can ask about the behaviour of $C(t)$ as t goes to infinity. By using Theorem 5.1.11 and its notation, for a positive function f,*

$$\int P_t(f)\Gamma(\ln P_tf)d\mu\leq B(t)\int fd\mu.$$

By taking f an approximation of the unity, we obtain:

$$C(t)\leq B(t).$$

And so for big t, $C(t)$ is less than a constant we can compute.

Remark 6.2.11. *In fact, it is possible to obtain a whole family of reverse poincaré inequality which interpoles between 6.1.1 and 6.2.1: Let $1<p\leq 2$ and $f:\mathbb{G}\to\mathbb{R}$ be a smooth function, then*

$$\Gamma(P_tf,P_tf)\leq -\frac{1}{2p}\frac{\partial}{\partial t}\int_{\mathbb{G}}\frac{p_t^{p-1}}{p-1}d\mu\int_{\mathbb{G}} f^2p_t^{2-p}d\mu.$$

Chapter 7

Subcommutation between the gradient and the semigroup

In this chapter, we are concerned by subcommutation inequalities between the gradient and the semigroup. More precisely, the inequalities that we deal with are the following:

$$\Gamma(P_t f) \leq C_2(t) P_t(\Gamma f) \tag{7.0.1}$$

and

$$\sqrt{\Gamma(P_t f)} \leq C_1(t) P_t(\sqrt{\Gamma f}) \tag{7.0.2}$$

for all smooth functions f and with $C_1(t)$ and $C_2(t)$ two positive functions which do not depend on the function f. Of course, the inequality (7.0.2) is stronger than (7.0.1). Indeed, inequality (7.0.2) with constant $C_1(t)$ implies (7.0.1) with constant $C_1(t)^2$. Therefore if $C_1(t)$ and $C_2(t)$ are the optimal constant in (7.0.2) and (7.0.1), than

$$C_2(t) \leq C_1(t)^2.$$

The plan of the chapter is the following. First we will recall the Riemannian setting and the relation between this kind of inequalities, the $CD(\rho, \infty)$ criterion and the lower bound of the Ricci curvature.
Then, we will derive the consequences in term of functional inequalities (7.0.2) and (7.0.1) in an abstract setting, that is for a general diffusion operator on a complete manifold. The consequences are the same as the ones obtained under the $CD(\rho, \infty)$ criterion: local Gross Log-Sobolev inequalities, Cheeger and Bobkov type isoperimetric inequalities...; and the way to obtain them is also the same. The difference is that, as in the general setting the function $C(t)$ is not necessarly continuous in 0, the equivalence between all these functional inequalities is not true anymore. We only have an implication.
Next, we will turn to our model spaces. First, we treat the Heisenberg group where the inequalities (7.0.1) and (7.0.2) hold with $C_2(t)$ and $C_1(t)$ constants stricly bigger than 1 for $t > 0$. They were respectively established by Driver and Melcher [39] and by H.Q. Li [70]. Note that inequality (7.0.1) also holds on general Lie groups (see [75]). Inequality (7.0.1) is relatively easy to obtain, we will include its proof for completness. Inequality 7.0.2 is much harder to obtain and we will give two new proofs ot this result; one based on a Cheeger type inequality and he other on a complex commutation between L and a complex gradient.

Finally, we will discuss what we can do on the spaces $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ by using the previous methods. For the moment, our results are not totally satisafactory. But, we will obtain however the inequality (7.0.1) on $\mathbf{SU}(2)$ with an exponential decay.

7.1 The Riemannian case

First we begin with a theorem for a general diffusion semigroup on a complete manifold M. We recall that the $CD(\rho,\infty)$ criterion reads

$$\forall f \in \mathcal{C}_c,\ \Gamma_2 f \geq \rho \Gamma f. \tag{7.1.3}$$

Then one has the well known theorem (see [9] for example).

Theorem 7.1.1. *For such a semigroup and $\rho \in \mathbb{R}$, the following propositions are equivalent.*

1. *the $CD(\rho,\infty)$ criterion holds*
2. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ \sqrt{\Gamma(P_t f)} \leq e^{-\rho t} P_t(\sqrt{\Gamma f})$
3. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ \Gamma(P_t f) \leq e^{-2\rho t} P_t(\Gamma f)$.

In fact, there are much more functional inequalities which are equivalent in this setting (see also [9, 4, 10]).

Theorem 7.1.2. *Let $(P_t)_{t\geq 0}$ be a semigroup as in Theorem 7.1.1 and $\rho \in \mathbb{R}$, the following propositions are equivalent.*

1. *the $CD(\rho,\infty)$ criterion holds*
2. *the Poincaré inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$:*
$$P_t(f^2) - P_t(f)^2 \leq \frac{1-e^{-2\rho t}}{\rho} P_t(\Gamma(f))$$
3. *the Beckner-Latała-Oleszkiewicz inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, f positive and $p \in (1,2]$:*
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \leq p\,\frac{1-e^{-2\rho t}}{2\rho}\, P_t(f^{p-2}\Gamma(f))$$
4. *the logarithmic Sobolev inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, f positive:*
$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \leq \frac{1-e^{-2\rho t}}{2\rho}\, P_t\left(\frac{\Gamma(f)}{f}\right)$$
5. *the reverse Poincaré inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$:*
$$P_t(f^2) - P_t(f)^2 \geq \frac{e^{2\rho t}-1}{\rho}\Gamma(P_t f)$$
6. *the reverse Beckner-Latała-Oleszkiewicz inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, f positive and $p \in (1,2]$:*
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \geq p\frac{e^{2\rho t}-1}{2\rho}(P_t f)^{p-2}\Gamma(P_t f)$$

7. *the reverse logarithmic Sobolev inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, f positive:*

$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \geq \frac{e^{2\rho t}-1}{2\rho}\frac{\Gamma(P_t f)}{P_t f}$$

8. *the Cheeger isoperimetric inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M})$, $\forall \mathbf{x} \in \mathcal{M}$:*

$$P_t(|f - P_t(f)(\mathbf{x})|)(\mathbf{x}) \leq 2\sqrt{\frac{1-e^{-2\rho t}}{\rho}} P_t(\sqrt{\Gamma}(f))(\mathbf{x})$$

9. *a first Bobkov isoperimetric inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$:*

$$\mathcal{I}(P_t f) - P_t(\mathcal{I} f) \leq \sqrt{\frac{1-e^{-2\rho t}}{\rho}} P_t(\sqrt{\Gamma} f)$$

10. *the Bobkov isoperimetric inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$:*

$$\mathcal{I}(P_t f) \leq P_t\left(\sqrt{\mathcal{I}(f)^2 + \frac{1-e^{-2\rho t}}{\rho}\Gamma(f)}\right)$$

11. *a first reverse Bobkov isoperimetric inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$:*

$$\mathcal{I}(P_t f) - P_t(\mathcal{I} f) \geq \sqrt{\frac{e^{2\rho t}-1}{\rho}} \sqrt{\Gamma P_t f}$$

12. *the reverse Bobkov isoperimetric inequality holds, $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}, (0,1))$:*

$$\mathcal{I}(P_t f) \geq \sqrt{(P_t\mathcal{I}(f))^2 + \frac{e^{2\rho t}-1}{\rho}\Gamma(P_t f)}.$$

where $\mathcal{I} : [0,1] \to [0,(2\pi)^{-1/2}]$ is the Gaussian isoperimetric function defined by $\mathcal{I} = (F_\gamma)' \circ (F_\gamma)^{-1}$ where

$$F_\gamma(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{1}{2}u^2}\, du.$$

Remark 7.1.3. *The proof of the direct sense of this theorem (i.e. the $CD(\rho,\infty)$ criterion implies all the other functional inequalities) is a direct corollary, except for the second Bobkov isoperimetric inequality and its reverse inequality, of the results of the next section which deals with the consequences of the subcommutation between the gradient and the semigroup in a general setting.*
The two Bobkov isoperimetric inequalities are in fact equivalent by a general argument of Barthe and Maurey [14] and therefore the second Bobkov isoperimetric inequality can be proven when only a subcommutation inequality between the gradient and the semigroup holds (but not by a direct argument). For the second reverse Bobkov inequality, we are not aware if such an inequality still holds under only a subcommutation inequality between the gradient and the semigroup.

Moreover, if M is a Riemannian manifold and if the generator of the semigroup is the Laplace-Beltrami operator on M, then one has $\Gamma(f) = |\nabla f|^2$ where ∇ is the gradient associated to the Riemannian metric on M. And in this setting, the propositions of the above theorem are equivalent to an uniform lower bound of the Ricci curvature on M. Then, one has (see [6, 68, 94]):

Theorem 7.1.4. *Let P_t be the semigroup associated to the Laplace-Beltrami operator of a complete Riemannian manifold M and $\rho \in \mathbb{R}$. The following propositions are equivalent.*

1. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \mathrm{Ricci}(\nabla f, \nabla f) \geq \rho\, |\nabla f|^2$
2. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \Gamma_2(f) \geq \rho\, |\nabla f|^2$
3. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ |\nabla P_t f|^2 \leq e^{-2\rho t}\, P_t(|\nabla f|^2)$
4. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ |\nabla P_t f| \leq e^{-\rho t}\, P_t(|\nabla f|)$

This is the case for some $\rho \in \mathbb{R}$ when $\mathcal{M}$ is compact. This is also the case with $\rho = 0$ when $\mathcal{M}$ is $\mathbb{R}^n$ equipped with the usual metric since Ricci $\equiv 0$. In this last example, L is the usual Laplace operator Δ and the explicit formula for the heat kernel gives $\nabla P_t f = P_t \nabla f$ for the usual gradient ∇ and thus $|\nabla P_t f| \leq P_t\, |\nabla f|$.

7.2 The consequences for a general diffusion semigroup

Most of the consequences of the classical gradient bounds under a Γ_2 curvature assumption remain true under an H.-Q. Li gradient bound. In the sequel, we derive, by interpolation from the gradient bound (7.0.2), several local functional inequalities of Gross-Poincaré-Cheeger-Bobkov type for the heat kernel on the Heisenberg group. The term *local* means that these inequalities concern the probability measure $P_t(\cdot)(\mathbf{x})$ at fixed t and $\mathbf{x}$, in contrast to inequalities for the invariant measure. In the literature, these inequalities and interpolations where mainly developed in Riemannian settings under a Γ_2 curvature assumption. Rigorously, the semigroup interpolations used in the sequel rely on the existence of an algebra of functions $\mathcal{A}$ from M to $\mathbb{R}$ stable by the action of the heat kernel. In all the section, we let L be a diffusion generator of a semi-group on a complete manifold M and we assume that L satisfies the inequality (7.0.2) with a general function $C_1(t)$.

7.2.1 Gross-Poincaré type inequalities

One of the first consequence of the gradient bound (7.0.2) is Gross-Poincaré type local inequalities, also called φ-Sobolev inequalities in [30, 55]. Namely, let $\varphi : I \to \mathbb{R}$ be a smooth convex function defined on an open interval $I \subset \mathbb{R}$ such that $\varphi'' > 0$ on I and $-1/\varphi''$ is convex on I.

Lemma 7.2.1. *For such a function φ, the bivariate function*

$$(u, v) : I \times \mathbb{R} \to \varphi''(u)v^2 \in \mathbb{R}$$

is convex.

Proof. To see this, note that the Hessian of the bivariate function writes:

$$\begin{pmatrix} \varphi^{(4)}(u)v^2 & 2\varphi^{(3)}(u)v^2 \\ 2\varphi^{(3)}(u)v^2 & 2\varphi''(u) \end{pmatrix}.$$

Note also that

$$\left(\frac{-1}{\varphi''(u)}\right)'' = \frac{\varphi''(u)\varphi^{(4)}(u) - 2\varphi^{(3)}(u)^2}{\varphi''(u)^3}.$$

The hypothesis on φ imply that the last quantity is non negative and as a by product that $\varphi^{(4)}(u)$ is also non negative. Consequently, the determinant and the trace of the Hessian are both non negative and the bivariate function is convex. □

Remark 7.2.2. *The following functions satisfy the above hyptothesis:*

- $\varphi(u) = u\ln u$ *on* $I = (0,\infty)$ *and* $\varphi''(u) = \frac{1}{u}$
- $\varphi(u) = u^p$, $1 < p \leq 2$ *on* $I = (0,\infty)$ *and* $\varphi''(u) = p(p-1)\frac{1}{u^{2-p}}$
- $\varphi(u) = u^2$ *on* $I = \mathbb{R}$ *and* $\varphi''(u) = 2$.

Now we can state the main results of this part.

Theorem 7.2.3 (Local Gross-Poincaré inequalities)**.** *For every $t \geq 0$, every $\mathbf{x} \in M$, and every $f \in \mathcal{C}_c^\infty(M, I)$,*

$$P_t(\varphi(f)) - \varphi(P_t f) \leq \left(\int_0^t C_1(u)^2 du\right) P_t\left(\varphi''(f)|\nabla f|^2\right) \tag{7.2.4}$$

Proof. One can assume that the support of f is strictly included in I. Since L is a diffusion operator, $L(\alpha(f)) = \alpha'(f)Lf + \alpha''(f)\Gamma f$ for any $f \in \mathcal{C}_c^\infty(M, \mathbb{R})$ and any smooth $\alpha : \mathbb{R} \to \mathbb{R}$. By the semigroup and the diffusion properties,

$$\begin{aligned} P_t(\varphi(f)) - \varphi(P_t f) &= \int_0^t \partial_s P_s(\varphi(P_{t-s}f))\, ds \\ &= \int_0^t P_s\left(L(\varphi(P_{t-s}f))\right) ds - \int_0^t P_s(\varphi'(P_{t-s}f)LP_{t-s}f)\, ds \\ &= \int_0^t P_s(\varphi''(P_{t-s}f)\Gamma(P_{t-s}f))\, ds. \end{aligned}$$

Now, (7.0.2) gives

$$\Gamma(P_{t-s}f) \leq C_1(t-s)^2 (P_{t-s}(\sqrt{\Gamma f}))^2.$$

Next, by the Cauchy-Schwarz inequality or alternatively by the Jensen inequality for the bivariate convex function $(u,v) \mapsto \varphi''(u)v^2$, we get

$$\varphi''(P_{t-s}f)(P_{t-s}(\sqrt{\Gamma f}))^2 \leq P_{t-s}(\varphi''(f)\Gamma(f)),$$

which gives the desired result, since then

$$P_t(\varphi(f)) - \varphi(P_t f) \leq \int_0^t C_1(t-s)^2 P_t(\varphi''(f)\Gamma(f)) ds.$$

□

- for $\varphi(u) = u\log(u)$ on $I = (0,\infty)$, we get a Gross logarithmic Sobolev inequality, mentioned for instance in [70] (see also [49, 50]),

$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \leq \left(\int_0^t C_1(u)^2 du\right) P_t\left(\frac{\Gamma(f)}{f}\right); \qquad (7.2.5)$$

- for $\varphi(u) = u^p$ on $I = (0,\infty)$ with $1 < p \leq 2$, we get a Beckner-Latała-Oleszkiewicz type inequality (see [20, 63])

$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \leq p\left(\int_0^t C_1(u)^2 du\right) P_t(f^{p-2}\Gamma(f)); \qquad (7.2.6)$$

- for $\varphi(u) = u^2$ on $I = \mathbb{R}$, we get a Poincaré inequality,

$$P_t(f^2) - (P_t(f))^2 \leq 2\left(\int_0^t C_1(u)^2 du\right) P_t(\Gamma(f)). \qquad (7.2.7)$$

We can now obtain some reverse inequalities.

Theorem 7.2.4 (Local Reverse Gross-Poincaré inequalities)**.** *For every $t \geq 0$, every $\mathbf{x} \in M$, and every $f \in \mathcal{C}_c^\infty(M, I)$,*

$$P_t(\varphi(f)) - \varphi(P_tf) \geq \left(\int_0^t \frac{1}{C_1(u)^2} du\right) \left(\varphi''(P_tf)\right) \Gamma(P_tf) \qquad (7.2.8)$$

Proof. As before, one can assume that the support of f is strictly included in I. And by the semigroup and the diffusion properties, one has

$$\begin{aligned} P_t(\varphi(f)) - \varphi(P_tf) &= \int_0^t \partial_s P_s(\varphi(P_{t-s}f))\, ds \\ &= \int_0^t P_s(\varphi''(P_{t-s}f)\Gamma(P_{t-s}f))\, ds. \end{aligned}$$

But with $g = P_{t-s}f$, by Cauchy-Schwartz inequality since $\varphi'' > 0$,

$$\left(P_s\left(\sqrt{\Gamma(g)}\right)\right)^2 \leq P_s(\Gamma(g)\varphi''(g))P_s\left(\frac{1}{\varphi''(g)}\right)$$

By hypothesis $\frac{1}{\varphi''}$ is concave, so

$$P_s\left(\frac{1}{\varphi''(g)}\right) \leq \frac{1}{\varphi''(P_sg)} = \frac{1}{\varphi''(P_tf)}.$$

Therefore

$$P_s(\Gamma(g)\varphi''(g)) \geq \varphi''(P_tf)\left(P_s\left(\sqrt{\Gamma(g)}\right)\right)^2.$$

Now, (7.0.2) gives

$$\left(P_s\left(\sqrt{\Gamma(g)}\right)\right)^2 \geq \frac{1}{C_1(s)^2}\Gamma(P_tf).$$

which ends the proof. □

- for $\varphi(u) = u\log(u)$ on $I = (0, \infty)$, we get a reverse Gross logarithmic Sobolev inequality,
$$P_t(f\log(f)) - P_t(f)\log(P_t(f)) \geq \left(\int_0^t \frac{1}{C_1(u)^2}du\right)\frac{\Gamma(P_t f)}{P_t f}; \tag{7.2.9}$$
- for $\varphi(u) = u^p$ on $I = (0, \infty)$ with $1 < p \leq 2$, we get a Reverse Beckner-Latała-Oleszkiewicz type inequality
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \geq p\left(\int_0^t \frac{1}{C_1(u)^2}du\right)(P_t f)^{p-2}\Gamma(P_t f); \tag{7.2.10}$$
- for $\varphi(u) = u^2$ on $I = \mathbb{R}$, we get a reverse Poincaré inequality,
$$P_t(f^2) - (P_t(f))^2 \geq 2\left(\int_0^t \frac{1}{C_1(u)^2}du\right)\Gamma(P_t f). \tag{7.2.11}$$

For the moment we deal with the consequences of the stronger inequality (7.0.2). But one can ask about the consequences of the a priori weaker inequality (7.0.1). For the Riemannian manifold, as we saw, it is known that the two inequalities are equivalent with $C_2 = C_1^2 = e^{-2\rho t}$ for some $\rho \in \mathbb{R}$. In a more general setting like subriemannian manifold, this is an open question. In this weaker case, the local Poincaré and reverse Poincaré inequalities are still valid.

Theorem 7.2.5. *Let L be a diffusion generator of a semigroup on a complete manifold M. Assume that L satisfies the inequality (7.0.1) with a general function $C_2(t)$. Then the following local Poincaré and reverse Poincaré inequalities hold:*
$$P_t(f^2) - (P_t f)^2 \leq 2\left(\int_0^t C_2(u)du\right)P_t(\Gamma(f)) \tag{7.2.12}$$
and
$$P_t(f^2) - (P_t f)^2 \geq 2\left(\int_0^t \frac{1}{C_2(u)}du\right)\Gamma(P_t f) \tag{7.2.13}$$
for all smooth f on M.

Proof. The proof is closed to the above ones. We work only with $\varphi(u) = u^2$ for which $\varphi''(u) = 2$. As before, we write:
$$\begin{aligned} P_t(f^2) - (P_t f)^2 &= \int_0^t \partial_s P_s\left((P_{t-s}f)^2\right) ds \\ &= 2\int_0^t P_s(\Gamma(P_{t-s}f))\, ds. \end{aligned}$$
And now we can use (7.0.1) in both sense to obtain the desired local Poincaré and reverse Poincaré inequalities. □

Remark 7.2.6. *We have seen that in the model spaces we consider, we have obtain the reverse Poincaré inequality with the optimal constants.*

The local reverse Poincaré inequality has the following direct consequence: (7.2.7) or (7.2.13)

Corollary 7.2.7. *Assume that (7.2.7) (respectively (7.2.13) holds. Then for all $t > 0$ and all $f \in \mathcal{C}_c^\infty(M, \mathbb{R})$,*
$$||\sqrt{\Gamma P_t f}||_\infty \leq a(t)||f||_\infty \tag{7.2.14}$$
with $a(t) = \frac{1}{\left(\int_0^t \frac{1}{C_1(u)^2}du\right)^{1/2}}$ (respectively $\frac{1}{\left(\int_0^t \frac{1}{C_2(u)}du\right)^{1/2}}$).

7.2.2 Cheeger type isoperimetric inequalities

Cheeger derived in [32] a lower bound for the spectral gap of the Laplacian on a Riemannian manifold. This bound can be related to a sort of L^1 Poincaré inequality, which has an isoperimetric content, see [34] and references therein. Here we derive such an inequality for the heat kernel by only using the gradient bound (7.0.2), by mixing arguments borrowed from [10] and [66].
The argument of the proof uses a consequence of a local reverse Poincaré inequality As done above, it is possible to deduce a reverse local Poincaré inequality from the gradient bounds (7.0.1) of Driver and Melcher or (7.0.2) of H.-Q. Li. However, as we will saw, the constants are not known precisely. As noticed above, a better constant (the optimal) is provided on our model spaces by theorems 6.2.1: For $t > 0$ and $g \in \mathbb{G}$,

$$\Gamma(P_t f, P_t f)(g) \le C(t)\left(P_t f^2(g) - (P_t f)^2(g)\right)$$

where

$$C(t) = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{G}} p_t \ln p_t d\mu.$$

Here we choose to work only with the Poincaré inequality obtained from 7.0.2) with a general function $C_1(t)$. Of course, when a better Poincaré inequality is known, one can use it instead.

Theorem 7.2.8 (Local Cheeger type inequality)**.** *With the notations of* (7.0.2)*, for every $t \ge 0$, every $\mathbf{x} \in M$, and every $f \in \mathcal{C}_c^\infty(M, \mathbb{R})$,*

$$P_t(|f - P_t(f)(\mathbf{x})|)(\mathbf{x}) \le 2R(t)\, P_t(\sqrt{\Gamma(f)})(\mathbf{x}). \tag{7.2.15}$$

where $R(t)$ is defined by

$$R(t) = \int_0^t C_1(s)\left(\int_0^s \frac{2}{C_1(u)^2}\,du\right)^{-\frac{1}{2}} ds.$$

Proof. We adapt the method used in [66, p. 953] for the invariant measure in Riemannian settings. For any $g \in \mathcal{C}_c^\infty(M, \mathbb{R})$ with $\|g\|_\infty \le 1$, any $t \ge 0$, and any $\mathbf{x} \in M$,

$$\begin{aligned}
P_t((f - P_t(f)(\mathbf{x}))g)(\mathbf{x}) &= P_t(fg)(\mathbf{x}) - P_t(f)(\mathbf{x})P_t(g)(\mathbf{x}) \\
&= \int_0^t \partial_s P_s((P_{t-s}f)(P_{t-s}g))(\mathbf{x})\,ds \\
&= 2\int_0^t P_s(\Gamma(P_{t-s}f, P_{t-s}g))(\mathbf{x})\,ds \\
&\le 2\int_0^t P_s(\sqrt{\Gamma(P_{t-s}f)}\sqrt{\Gamma(P_{t-s}g)})(\mathbf{x})\,ds \\
&\le 2\int_0^t a(t-s)C_1(t-s)ds\, P_t(\sqrt{(\Gamma(f))})(\mathbf{x})\,\|g\|_\infty .
\end{aligned}$$

where we used the gradient bound (7.0.2) for f and the gradient bound (7.2.14) for g. The desired result follows then by $\mathrm{L}^1 - \mathrm{L}^\infty$ duality by taking the supremum over g and using the explicit value of a. □

Similarly, we get also the following correlation bound for every $t \geq 0$ and $f, g \in \mathcal{C}_c^\infty(M, \mathbb{R})$,

$$|P_t(fg) - P_t(f)P_t(g)| \leq 2\int_0^t C_1(u)^2 du \sqrt{P_t(\Gamma(f))}\sqrt{P_t(\Gamma(g))}. \tag{7.2.16}$$

When $f = g$, we recover the Poincaré inequality (7.2.7).

Theorem 7.2.9 (Yet another local Cheeger type inequality)**.** *With the notations of* (7.0.2)*, let $t \geq 0$, $\mathbf{x} \in M$, and B be a Borel subset of M, there exists a real constant $C_{B,t,\mathbf{x}} > 1$ such that for every function $f \in \mathcal{C}_c^\infty(M, \mathbb{R})$ which vanishes on B,*

$$|P_t(f)(\mathbf{x})| \leq C_{B,t,\mathbf{x}} \, P_t(|\nabla f|)(\mathbf{x}). \tag{7.2.17}$$

Proof. Let $g \in \mathcal{C}^\infty(M, \mathbb{R})$ be such that $\|g\|_\infty < \infty$ and $g \equiv 1$ on B^c. Since $fg = f$, the computation made in the proof of theorem 7.2.8 provides

$$P_t(f)(\mathbf{x}) - P_t(f)(\mathbf{x})P_t(g)(\mathbf{x}) \leq 2R(t) \, \|g\|_\infty \, P_t(|\nabla f|)(\mathbf{x}).$$

For any arbitrary real number $r \geq 1$, the class of functions

$$\mathcal{C}_{B,r} = \{g \in \mathcal{C}^\infty(M, \mathbb{R}) \text{ with } \|g\|_\infty \leq r \text{ and } g \equiv 1 \text{ on } B^c\}.$$

is not empty since it contains the constant function $\equiv 1$. Furthermore, since $P_t(\cdot)(\mathbf{x})$ is a probability measure with non vanishing density, the following extrema

$$\alpha_-(B, r, t, \mathbf{x}) = \inf_{g \in \mathcal{C}_{B,r}} P_t(g)(\mathbf{x}) \quad \text{and} \quad \alpha_+(B, r, t, \mathbf{x}) = \sup_{g \in \mathcal{C}_{B,r}} P_t(g)(\mathbf{x})$$

are finite and non zero. Moreover, an elementary local perturbative argument on any element of the class $\mathcal{C}_{B,r}$ shows that $\alpha_-(B, r, t, \mathbf{x}) \, \alpha_+(B, r, t, \mathbf{x}) < 0$ as soon as r is large enough, say $r \geq r_{B,t,\mathbf{x}}$. Thus, $P_t(f)(\mathbf{x})P_t(g)(\mathbf{x}) \leq 0$ for some $g \in \mathcal{C}_{B,r}$. The desired result follows then with $C_{B,t,\mathbf{x}} = 2R(t) \, r_{B,t,\mathbf{x}}$, since one can replace f by $-f$ in the obtained inequality. Note that $C_{B,t,\mathbf{x}}$ blows up when $\mathrm{vol}(B) \searrow 0$. □

The isoperimetric content of (7.2.15) can be extracted by approximating an indicator with a smooth f, see for instance [10]. Namely, for any Borel set $A \subset M$ with smooth boundary, any $t \geq 0$, and any $\mathbf{x} \in M$, we get by denoting $\mu_{t,\mathbf{x}} = P_t(\cdot)(\mathbf{x})$,

$$\mu_{t,\mathbf{x}}(A)(1 - \mu_{t,\mathbf{x}}(A)) \leq R(t) \, \mu_{t,\mathbf{x}}^{\text{surface}}(\partial A) \tag{7.2.18}$$

where $\mu_{t,\mathbf{x}}^{\text{surface}}(\partial A)$ is the perimeter of A for $\mu_{t,\mathbf{x}}$ as defined in [3, Section 3] (see also [78]). From (7.2.17), we get similarly for any Borel set B in M and any Borel set $A \subset B^c$ with smooth boundary,

$$\mu_{t,\mathbf{x}}(A) \leq C_{B,t,\mathbf{x}} \, \mu_{t,\mathbf{x}}^{\text{surface}}(\partial A). \tag{7.2.19}$$

7.2.3 Bobkov type isoperimetric inequalities

Let $F_\gamma : \mathbb{R} \to [0, 1]$ be the cumulative probability function of the standard Gaussian distribution γ on the real line $\mathbb{R}$, given for every $t \in \mathbb{R}$ by

$$F_\gamma(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{1}{2}u^2} \, du.$$

The Gaussian isoperimetric function $\mathcal{I} : [0,1] \to [0,(2\pi)^{-1/2}]$ is defined by $\mathcal{I} = (F_\gamma)' \circ (F_\gamma)^{-1}$. The function $\mathcal{I}$ is concave, continuous on $[0,1]$, smooth on $(0,1)$, symmetric with respect to the vertical axis of equation $u = 1/2$, and satisfies to the differential equation

$$\mathcal{I}(u)\mathcal{I}''(u) = -1 \quad \text{for any } u \in [0,1] \tag{7.2.20}$$

with $\mathcal{I}(0) = \mathcal{I}(1) = 0$ and $\mathcal{I}'(0) = -\mathcal{I}'(1) = \infty$. Note that $\mathcal{I}(u) \geq u(1-u)$ for any real $u \in [0,1]$, and that $\mathcal{I}(u) \leq \min(u, 1-u))$ when u belongs to a neighborhood of $1/2$.

Lemma 7.2.10 (Yet another uniform gradient bound)**.** *With the notations of* (7.0.2)*, for every* $t \geq 0$ *and* $f \in \mathcal{C}_c^\infty(M,(0,1))$*,*

$$\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) \leq R(t)\, P_t(\sqrt{\Gamma f}). \tag{7.2.21}$$

Proof. The inequality (7.2.21) was obtained by Bobkov in [22] for the standard Gaussian measure on $\mathbb{R}$. Later, it was generalized in [10], by using semigroup techniques, to Riemannian settings under a Γ_2 curvature assumption. We give here a proof by adapting the argument given in [10, p. 261-263] from invariant measure settings to local settings. One may assume that $\varepsilon \leq f \leq 1-\varepsilon$ for some $\varepsilon > 0$. By the diffusion property and (7.2.20)

$$\begin{aligned}
[\mathcal{I}(P_t f)]^2 - [P_t(\mathcal{I}(f))]^2 &= -\int_0^t \partial_s [P_s(\mathcal{I}(P_{t-s}f))]^2 \, ds \\
&= -2\int_0^t P_s(\mathcal{I}(P_{t-s}f)) P_s\left(\mathcal{I}''(P_{t-s}f)\Gamma(P_{t-s}f)\right) ds \\
&= +2\int_0^t P_s(\mathcal{I}(P_{t-s}f)) P_s\left(\frac{\Gamma(P_{t-s}f)}{\mathcal{I}(P_{t-s}f)}\right) ds.
\end{aligned}$$

Next, the Cauchy-Schwarz inequality or alternatively the Jensen inequality for the bivariate convex function $(u,v) \mapsto u^2/\mathcal{I}(v) = -\mathcal{I}''(v)u^2$ gives

$$[\mathcal{I}(P_t f)]^2 - [P_t(\mathcal{I}(f))]^2 \geq 2\int_0^t \left[P_s(\sqrt{\Gamma(P_{t-s}f)})\right]^2 ds.$$

Now by using the gradient bound (7.0.2) we have

$$C_1(s)\, P_s(\sqrt{\Gamma(P_{t-s}f)}) \geq \sqrt{\Gamma(P_s(P_{t-s}f))} = \sqrt{\Gamma(P_t f)}$$

and thus

$$[\mathcal{I}(P_t f)]^2 - [P_t(\mathcal{I}(f))]^2 \geq 2\int_0^t \frac{1}{C_1(u)^2} du\, \Gamma(P_t f).$$

In particular, we obtain the following uniform gradient bound

$$\left\|\mathcal{I}''(P_t f)\sqrt{\Gamma(P_t f)}\right\|_\infty = \left\|\frac{\sqrt{\Gamma(P_t f)}}{\mathcal{I}(P_t f)}\right\|_\infty \leq \left(2\int_0^t \frac{1}{C_1(u)^2} du\right)^{-1/2}.$$

We are now able to prove (7.2.21). By the diffusion property

$$\begin{aligned}
\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) &= -\int_0^t \partial_s\, P_s(\mathcal{I}(P_{t-s}f))\, ds \\
&= -\int_0^t P_s(\mathcal{I}''(P_{t-s}f)\Gamma(P_{t-s}f))\, ds.
\end{aligned}$$

By (7.0.2) we get

$$\Gamma(P_{t-s}f) \leq C_1(t-s)\sqrt{\Gamma(P_{t-s}f)}P_{t-s}(\sqrt{\Gamma(f)})$$

and thus

$$\mathcal{I}(P_tf) - P_t(\mathcal{I}(f)) \leq R(t)\,P_t(|\nabla f|).$$

□

The isoperimetric content of (7.2.21) can be extracted by approximating an indicator with a smooth f, see [10]. Namely, for any Borel set $A \subset M$ with smooth boundary, any $t \geq 0$, and any $\mathbf{x} \in \mathbb{H}$, we get by denoting $\mu_{t,\mathbf{x}} = P_t(\cdot)(\mathbf{x})$,

$$\mathcal{I}(\mu_{t,\mathbf{x}}(A)) \leq C_1^2\sqrt{2t}\,\mu_{t,\mathbf{x}}^{\mathrm{surface}}(\partial A). \tag{7.2.22}$$

Corollary 7.2.11 (Yet another local Bobkov Gaussian isoperimetric inequality)**.** *With the notations of* (7.0.2)*, for every $t \geq 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{H},(0,1))$,*

$$\mathcal{I}(P_tf) \leq P_t\left(\sqrt{(\mathcal{I}(f))^2 + R(t)^2\,|\nabla f|^2}\right). \tag{7.2.23}$$

Proof. The desired result follows from the transportation-rearrangement argument given in [14, prop. 5 p. 427], which is inspired from [10, p. 273]. The method is not specific to the heat semigroup on our particular subelliptic structures. It is based in particular on a similar inequality for the standard Gaussian measure on $\mathbb{R}$ obtained by Bobkov in [23]. □

One of the most important aspect of (7.2.23) is its stability by tensor product, in contrast with (7.2.21), while maintaining the same isoperimetric content. Moreover, one may recover from (7.2.23) the Gross logarithmic Sobolev inequality (7.2.5) by using the fact that $\mathcal{I}'(u) \sim \sqrt{-2\log(u)}$ and $\mathcal{I}(u) \sim u\sqrt{-2\log(u)}$ at $u = 0$. We ignore if (7.2.23) can be obtained directly by semigroup interpolation, as for the elliptic case in [10]. The proof given in [10] for the elliptic case is based directly on a curvature bound at the level of the infinitesimal generator, which is not implied by the gradient bound (7.0.2). We ignore also if one can adapt in our subelliptic setting the two points space approach used in [23] or the martingale representation approach used in [14, 29, 54, 68]. There is a lack of a direct proof of (7.2.23) in our subelliptic setting (even on the Heisenberg group), despite the fact that (7.2.23) and (7.2.21) are equivalent, according to the argument of Barthe and Maurey in [14, prop. 5 p. 427].
We can also prove some reverse form of these inequalities.

Lemma 7.2.12 (A first reverse inequality)**.** *With the notations of* (7.0.2)*, for every $t \geq 0$ and $f \in \mathcal{C}_c^\infty(M,(0,1))$,*

$$\mathcal{I}(P_tf) - P_t(\mathcal{I}(f)) \geq r(t)\sqrt{\Gamma(P_tf)}. \tag{7.2.24}$$

with

$$r(t) = \int_0^t \left(2\int_0^s C_1(u)^2du\right)^{-1/2}\frac{1}{C_1(s)}ds$$

Proof. As before, we write,

$$[\mathcal{I}(P_tf)]^2 - [P_t(\mathcal{I}(f))]^2 = +2\int_0^t P_s(\mathcal{I}(P_{t-s}f))P_s\left(\frac{\Gamma(P_{t-s}f)}{\mathcal{I}(P_{t-s}f)}\right)ds.$$

Next, the Jensen inequality for the bivariate convex function $(u,v) \mapsto u^2/\mathcal{I}(v) = -\mathcal{I}''(v)u^2$ gives

$$P_s\left(\frac{\Gamma(g)}{\mathcal{I}(g)}\right) \leq \frac{(P_s(\sqrt{\Gamma(g)}))^2}{\mathcal{I}(P_sg)}$$

for all smooth function g. We use it with $g = P_{t-s}f$ and by using the gradient bound (7.0.2) we have

$$P_s(\sqrt{\Gamma(g)}) = P_s(\sqrt{\Gamma(P_{t-s}f)}) \leq C_1(t-s)P_s(P_{t-s}\sqrt{\Gamma(f)}) = C_1(t-s)P_t(\sqrt{\Gamma(f)}).$$

Therefore,

$$[\mathcal{I}(P_t f)]^2 - [P_t(\mathcal{I}(f))]^2 \leq 2\int_0^t C_1(u)^2 du\, P_t(\Gamma f).$$

In particular, we obtain the following uniform gradient bound

$$-\mathcal{I}''(P_t f)P_t(\sqrt{\Gamma(f)}) \geq \left(2\int_0^t C_1(u)^2 du\right)^{-1/2}.$$

As before, we write:

$$\begin{aligned}
\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) &= -\int_0^t P_s(\mathcal{I}''(P_{t-s}f)\Gamma(P_{t-s}f))\, ds. \\
&\geq \int_0^t \left(2\int_0^{t-s} C_1(u)^2 du\right)^{-1/2} P_s(\sqrt{\Gamma P_{t-s}f}) ds \\
&\geq \int_0^t \left(2\int_0^{t-s} C_1(u)^2 du\right)^{-1/2} \frac{1}{C_1(t-s)} ds \sqrt{\Gamma(P_t f)}.
\end{aligned}$$

The change of variables: s gives $t-s$ ends the proof. □

7.3 The Heisenberg group

7.3.1 The Driver-Melcher inequality

We give here an elementary proof of the Driver and Melcher gradient bound (7.0.1). The argument is simply an integration by parts followed by the upper bound on $\Gamma(\log h_1, \log h_1)$ obtained in Section 4.6.2. Indeed, from the inequalities (4.6.13) and (4.6.14), it is quite clear that the constant

$$A = \int \|\mathbf{x}\|\Gamma(\log h_1, \log h_1)(\mathbf{x})h_1(\mathbf{x})d\mathbf{x} \tag{7.3.25}$$

is finite, where $\|\mathbf{x}\|$ denotes as usual the Euclidean norm of the horizontal projection of the point $\mathbf{x}$. Then, we have the following theorem.

Theorem 7.3.1. *There exits a constant $C_2 > 1$ such that for every $t \geq 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{H})$,*

$$\Gamma(P_t f, P_t f) \leq C_2 P_t(\Gamma(f, f)).$$

Moreover the constant C_2 can be chosen to be $C_2 = 2(A+4)$ with A the constant defined by (7.3.25).

Proof. We assume that $\mathbf{x} = 0$ (by group action) and $t = 1$ (by dilation). Then, we write

$$XP_1 f(0) = P_1(\hat{X}f)(0) = \int (X + 2yZ)fh_1 dx.$$

An integration by parts for $\int 2yZ(f)\, h_1 dx = \int y(XY - YX)(f)\, h_1 dx$ gives

$$\int X(f)\,(yY(\log h_1) + 1)\, h_1 dx - \int Y(f)\, yX(\log h_1)\, h_1 dx$$

and a similar formula holds for $YP_1 f$. Next, we take a vector $(a, b) \in \mathbb{R}^2$ of unit norm and we use the Cauchy-Schwarz inequality to get

$$(aXP_1(f)(0) + bYP_1(f)(0))^2 \leq P_1(X(f)^2)A_1 + P_1(Y(f)^2)A_2$$

where

$$A_1 = P_1[((yY(\log h_1) + 2)a - xY(\log h_1)b)^2]$$

and

$$A_2 = P_1[((xX(\log h_1) + 2)b - yX(\log h_1)a)^2].$$

The desired inequality comes then from the upper bound

$$\max(A_1, A_2) \leq A_1 + A_2 \leq 2(A + 4).$$

Note that the obtained constant $C_2 = 2(A + 4)$ is certainly not the optimal one. □

Remark 7.3.2. *Since now, we denote by C_2 the optimal constant in Theorem 7.3.1. Of course we have $C_2 > 1$. Indeed the case $C_2 = 1$ would imply the $CD(0, \infty)$ criterion. Actually, using explicit polyomials, it is shown in [39] that $C_2 \geq 2$.*
As mentionned before, inequality of Theorem 7.3.1 implies the reverse Poincaré inequality; and it would imply the optimal reverse Poincaré inequality only if $C_2 = 2$. (This fact gives also another proof that $C_2 \geq 2$).

Remark 7.3.3. *The same method extends for all $p > 1$ and one gets, for $p > 1$, there exists a constant $C_p > 1$ such that for any smooth $f : \mathbb{H} \to \mathbb{R}$ and any $g \in \mathbb{H}$*

$$\sqrt{\Gamma(P_t f, P_t f)(g)} \leq C_p \left(P_t \Gamma(f, f)^{\frac{p}{2}}(g)\right)^{\frac{1}{p}}, \quad t \geq 0.$$

But this method fails for $p = 1$ since the quantities like $yY(\log h_1)$ are not bounded above. This is also the case for the method used in [39].

7.3.2 The H.Q. Li inequality via a Cheeger inequality

We propose two alternate and independent proofs of the H.-Q. Li inequality (7.0.2). This section is dedicated to the first proof. The first proof uses some basic symmetry considerations and a particular case of the Cheeger inequality of theorem 7.2.9 that we have to show by hands. The second proof relies on an explicit commutation between the complex gradient and the heat semigroup. Both mainly rely on the previous sharp estimates on the heat kernel that were obtained in [19]. First we state the H.Q. Li inequality.

Theorem 7.3.4. *There exits a constant $C_1 > 1$ such that for every $t \geq 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{H})$,*

$$\sqrt{\Gamma(P_t f, P_t f)} \leq C_1 P_t(\sqrt{\Gamma(f, f)}).$$

Remark 7.3.5. *Actually, from Remark 7.3.2, the best constant* C_1 *in Theorem 7.3.4 satisfies* $C_1 \geq \sqrt{2}$. *As mentionned before, inequality of Theorem 7.3.1 would imply the optimal reverse Poincaré inequality only if* $C_1 = \sqrt{2}$. *We conjecture* $C_1^2 = C_2 = 2$.

The first step in the proof is the following Cheeger type lemma.

Lemma 7.3.6. *For any real* $R > 0$*, there exists a real constant* $C > 0$ *such that for any smooth* $f : \mathbb{H} \to \mathbb{R}$ *which vanishes on the ball centered at* 0 *and of radius* R *for the Carnot-Carathéodory distance, we have*

$$\left| \int f h d\mathbf{x} \right| \leq C \int |\nabla f| \, h d\mathbf{x}$$

where h *is as before the density of* $P_1(0, d\mathbf{x})$.

Before we do the proof, we explain a little the geodesics on $\mathbb{H}$ and the choice of geodesic coordinates we will use in the sequel. The geodesics on $\mathbb{H}$ are well known, one can consult [77] or also [61, 60]. By left invariance, it is enough to describe the ones starting from the identity. They can be described in the following way: in exponential coordinates (x, y, z), the two first coordinates (x, y) parametrizes an arc of circle whereas the z variable equals two times the area swept out by the arc of circle.
Here we choose to parametrize geodesics of the Heisenberg group by the center $u \in \mathbb{C}$ of the circle in the $\mathbb{C}$-plane and by l the length of the arc of circle. In this system of coordinates, geodesics starting from $0_{\mathbb{H}}$ are given by the curves

$$\gamma_u(l) = \left(u(1 - \exp(\frac{il}{|u|})), |u|^2 (\frac{l}{|u|} - \sin(\frac{l}{|u|})) \right) \tag{7.3.26}$$

for $u \in \mathbb{C}$, $s \in \mathbb{R}$ and $s \in [0, 2\pi|u|]$.
With these coordinates we obtain the strait line geodesics in the $\mathbb{C}$ plane by letting go u to infinity in a given direction. Indeed we get at the limit the line passing by $0_{\mathbb{H}}$ orthogonal to the u direction.
We put

$$M_s(u, l) = \gamma_u(sl)$$

In the following when $l = 1$ we will note $M_s(u, 1) = M_s(u)$. We have the following properties:

- $M_s(u, l) = M_1(u, sl)$.
- The projection of the curve $\gamma_u(l)$ on the plan $\mathbb{C}$ is a an arc of circle. The radius of the circle is $|u|$. The angle of the arc is $\frac{l}{|u|}$.
- At the time $s = 2\pi|u|$, the point $\gamma_u(s)$ is on the $(0, z)$ axis. And after this times the curves is no longer a geodesic. So a geodesic hits the 1-sphere if and only if $|u| \geq \frac{1}{2\pi}$.
- sl is the curvilign abscisse of the point $M_s(u, l)$ on the arc of circle. This correspond to the Carnot-Caratheodory distance between the origin and the point $\gamma_u(sl) \in \mathbb{H}$.
- The curve $s \to M_s(u, l)$ is a subriemannian geodesic with constant speed (equal to 1 when $l = 1$).
- The Euclidean distance at the origin in the plan $\mathbb{C}$ denoted by x_s is $2|u| \sin\left(\frac{sl}{2|u|}\right)$.

- As we claimed it, the height of our point $\gamma_u(sl)$ is given by two times the area between the arc of circle and the line D joigning the point at the origine $0_{\mathbb{H}}$. This area equals

$$\frac{|u|^2}{2}(\frac{sl}{|u|}-\sin(\frac{sl}{|u|})).$$

To see this, we compute the area by Fubini, integrating the length of the chords parallel to the line D. We do it first when the angle $\phi=\frac{sl}{|u|}\le\pi$, we then get with R the radius of the cercle and R_0 the distance between the center of the circle and the line D,

$$\begin{aligned} R^2\int_{\frac{R_0}{R}}^{1}2\sqrt{1-u^2}\,du &= 2R^2\int_0^{\arccos\left(\frac{R_0}{R}\right)}\sin^2 v\,dv \\ &= R^2\arccos\left(\frac{R_0}{R}\right)-\frac{\sin\left(2\arccos\left(\frac{R_0}{R}\right)\right)}{2} \end{aligned}$$

and we conclude by noticing $\frac{R_0}{R}=\cos\frac{\phi}{2}$. For $\phi\ge\pi$, we compute the area by

$$\mathbb{R}^2\left[\pi-\frac{1}{2}\big((2\pi-\phi)-\sin(2\pi-\phi)\big)\right]$$

and we see the first formula is still valid.

- For all $s>1$, M_s is a diffeomoprhism between $\{(u,l)\in\mathbb{C}\times\mathbb{R}_+, l=1, \frac{s}{2\pi}<|u|\}$ and the s sphere of $\mathbb{H}$ without the two points which belong to the $(0,z)$ axis and the circle which is the intersection between the sphere and the $\mathbb{C}$ plane. So we can write

$$\int_{\mathbb{H}_3-B} fh d\lambda=\int_{|u|\ge\frac{1}{2\pi}} f(M_s(u))h(M_s(u))|\mathrm{Jac}(M_s)(u)|duds$$

We can then calculate the Jacobian of our diffeomorphism. We thank Nathaniel Eldredge for pointing us a small mistake in the computation done in [7]. Actually the mistake does not change the asymptotics of the function and the overall correctness of [7] is not affected. We also refer to [61, 60] where such a computation was done with slightly different coordinates. Here, we have for all $l\le 2\pi|u|$

$$\mathrm{Jac}(M_s)(u)=8|u|\sin\left(\frac{s}{2|u|}\right)\left(\sin\left(\frac{s}{2|u|}\right)-\frac{s}{2|u|}\cos\left(\frac{s}{2|u|}\right)\right). \tag{7.3.27}$$

Let us see this result. Our geodesic coordinates write

$$\big((1-\cos\phi)u_1+\sin\phi u_2, -\sin\phi u_1+(1-\cos\phi)u_1, (u_1^2+u_2^2)(\phi-\sin\phi)\big)$$

where $\phi=\frac{s}{|u|}$. The Jacobian is clearly invariant by rotation in the $\mathbb{C}$-plane, we can then compute it only at the points $u=(u_1,0)$. It is given by:

$$|\mathrm{Jac}(M_s)(u)|=\begin{vmatrix} 1-\cos\phi & \sin\phi & \sin\phi \\ -\sin\phi & 1-\cos\phi & -\cos\phi \\ 2u_1(\phi-\sin\phi) & 0 & u_1(1-\cos\phi) \end{vmatrix}$$

and thus

$$\begin{aligned}\mathrm{Jac}(M_s)(u) &= 4u_1(1-\cos\phi) - 2u_1\phi\sin\phi \\ &= 8u_1\sin\left(\frac{\phi}{2}\right)\left(\left(\frac{\phi}{2}\right) - \left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\right).\end{aligned}$$

Proof.[Proof of Lemma 7.3.6] For simplicity we work with $R=1$. Next, we make use of the polar coordinates which appear in (7.3.26). Namely, we parameterize the exterior of the unit ball by (u,s), with $u\in\mathbb{C}^2$, $|u|\geq\frac{1}{2\pi}$ and $s\in(1,2\pi\,|u|)$ for $z>0$ and $s\in(-2\pi\,|u|\,,-1)$ for $z<0$, with

$$(x+iy,z) = \left(u\left(1-\exp\left(\frac{is}{|u|}\right)\right), |u|^2\left(\frac{s}{|u|} - \sin\left(\frac{s}{|u|}\right)\right)\right). \tag{7.3.28}$$

Actually, for simplicity, we will work only in the half space $z>0$. The computations for the other half space $z<0$ are exactly the same. The unit ball (or more precisely the part of the unit ball included in the half space $z>0$ is the set $\{0\leq s\leq 1\}$, and since f is supported outside the unit ball, we write

$$|f(u,s)| = \left|\int_1^s \nabla f(u,t)\cdot e_t dt\right| \leq \int_1^s |\nabla f|\,(u,t)dt$$

where e_t is the unit vector along the geodesic. Let us write $A(u,t)dudt$ the Lebesgue measure on $\mathbb{R}^3$ in those coordinates. We write

$$\int |f(u,s)|\,h(u,s)A(u,s)duds \leq \int |\nabla f|\,(u,t)\left(\int_t^{2\pi|u|} A(u,s)h(u,s)ds\right)dudt$$

and we shall have proved our inequality when we have proved that

$$\int_t^{2\pi|u|} A(u,s)h(u,s)ds \leq CA(u,t)h(u,t), \tag{7.3.29}$$

for any (u,t) such that $|u|\geq\frac{1}{2\pi}$ and $t\geq 1$. In this computation, we forget the points in the (x,y) plane and the z-axis, but this is irrelevant since they have 0-measure. The computation of the Jacobian gives

$$A(u,s) = 8|u|\sin\left(\frac{s}{2|u|}\right)\left(\sin\left(\frac{s}{2|u|}\right) - \frac{s}{2|u|}\cos\left(\frac{s}{2|u|}\right)\right)$$

and the estimate (4.6.13) shows that we may replace $h(u,s)$ by

$$\frac{\exp(-\frac{s^2}{4})}{\sqrt{1+2s\,|u|\sin(\frac{s}{2|u|})}}$$

since the Euclidean norm of the horizontal projection of the point whose coordinates are (u,s) is $2\,|u|\sin(\frac{s}{2|u|})$. Setting $\tau=\frac{s}{2|u|}$ and $r=|u|$, the question is therefore to check that, for some constant C, for any $r\geq\frac{1}{2\pi}$, for any $\tau_0\geq\frac{1}{2r}$, one has

$$r^2\int_{\tau_0}^{\pi}\frac{\sin\tau(\sin\tau-\tau\cos\tau)}{\sqrt{1+4r^2\tau\sin\tau}}e^{-\tau^2r^2}d\tau \leq Cr\frac{\sin\tau_0(\sin\tau_0-\tau_0\cos\tau_0)}{\sqrt{1+4r^2\tau_0\sin\tau_0}}e^{-\tau_0^2r^2}.$$

Up to some constant, we may replace $\sin\tau(\sin\tau - \tau\cos\tau)$ by τ^4 on $(0, \frac{\pi}{2})$ and by $\pi - \tau$ on $(\frac{\pi}{2}, \pi)$. In the same way, we may replace $\sqrt{1 + 4r^2\tau\sin\tau}$ by $r\tau$ when $\tau < \frac{\pi}{2}$ (since $r\tau \geq \frac{1}{2}$) and by $1 + r\sqrt{\pi - \tau}$ when $\tau \in (\frac{\pi}{2}, \pi)$.

To obtain the desired inequality, we will use the following fact:

$$\forall A_0 > 0, \forall A \geq A_0, \forall p \in R, \int_A^\infty s^p \exp(-s^2)ds \leq C_p A^{p-1}\exp(-A^2). \tag{7.3.30}$$

The proof of this fact can be obtained by an integration by parts. Let $I_p = \int_A^\infty s^p \exp(-s^2)ds$

$$I_p = \frac{1}{2}A^{p-1}\exp(-A^2) + \frac{p-2}{2}I_{p-2},$$

but

$$I_{p-2} \leq \frac{1}{A_0}I_p$$

and so if A_0 is big enough

$$I_p \leq \frac{1}{2}A^{p-1}\exp(-A^2) + \frac{1}{2}I_p.$$

This gives the conclusion if A_0 is big enough. The result for all A_0 is obtained by continuity. Of course the constant C_p explodes when A_0 goes to 0.

Now, we return to the proof of (7.3.29). We first consider the case where $\tau_0 < \frac{\pi}{2}$, by the previous estimates we have

$$r^2\int_{\tau_0}^{\pi} \frac{\sin\tau(\sin\tau - \tau\cos\tau)}{\sqrt{1 + 4r^2\tau\sin\tau}} e^{-\tau^2r^2}d\tau \leq C\int_{\tau_0}^{\pi} r\tau^3 e^{-\tau^2r^2}d\tau$$

for some constant C and we may replace

$$r\frac{\sin\tau_0(\sin\tau_0 - \tau_0\cos\tau_0)}{\sqrt{1 + 4r^2\tau_0\sin\tau_0}}e^{-\tau_0^2r^2} \text{ by } \tau_0^3 e^{-\tau_0^2r^2}.$$

By inequality (7.3.30), one has:

$$\begin{aligned}\int_{\tau_0}^{\pi} r\tau^3 e^{-\tau^2r^2}d\tau &\leq \int_{r\tau_0}^{\infty}\frac{u^3}{r^3}e^{-u^2}du \\ &\leq C_3\frac{\tau_0^2}{r}\end{aligned}$$

and we are done since $\tau_0 \geq \frac{1}{r}$.
When $\tau_0 > \frac{\pi}{2}$, one uses the same estimates and replace

$$r^2\int_{\tau_0}^{\pi}\frac{\sin\tau(\sin\tau - \tau\cos\tau)}{\sqrt{1 + 4r^2\tau\sin\tau}}e^{-\tau^2r^2}d\tau \text{ by } \int_{\tau_0}^{\pi}\frac{r^2(\pi - \tau)}{1 + r\sqrt{\pi - \tau}}e^{-\tau^2r^2}d\tau$$

and

$$r\frac{\sin\tau_0(\sin\tau_0 - \tau_0\cos\tau_0)}{\sqrt{1 + 4r^2\tau_0\sin\tau_0}}e^{-\tau_0^2r^2} \text{ by } \frac{r(\pi - \tau_0)}{1 + r\sqrt{\pi - \tau_0}}e^{-\tau_0^2r^2}.$$

Now

$$r^2 \int_{\tau_0}^{\pi} \frac{\sin\tau(\sin\tau - \tau\cos\tau)}{\sqrt{1+4r^2\tau\sin\tau}} e^{-\tau^2 r^2} d\tau \leq \frac{r(\pi-\tau_0)}{1+r\sqrt{\pi-\tau_0}} \int_{\tau_0}^{\pi} r e^{-\tau^2 r^2} d\tau$$

and by using (7.3.30)

$$\begin{aligned} \int_{\tau_0}^{\pi} r e^{-\tau^2 r^2} d\tau &\leq \int_{r\tau_0}^{\infty} e^{-u^2} du \\ &\leq \frac{C_0}{r\tau_0} e^{-\tau_0^2 r^2} \end{aligned}$$

which ends the proof since $r\tau_0 \leq \frac{1}{2}$.
Observe that the same reasoning on a ball of radius ϵ would provide a constant which goes to infinity when ϵ goes to 0, as for the usual heat kernel on $\mathbb{R}^d$. □

In fact, we shall also use a slightly improved version of lemma 7.3.6.

Lemma 7.3.7. *For every real $R > 0$, if B is the ball centered at 0 and of radius R for the Carnot-Carathéodory distance, there exists a real constant $C > 0$ such that for any smooth $f : \mathbb{H} \to \mathbb{R}$,*

$$\int_{B^c} \left| f - \frac{1}{m(B)} \int_B f \, d\mathbf{x} \right| h d\mathbf{x} \leq C \int |\nabla f| \, h d\mathbf{x}$$

where $B^c = \mathbb{H} \setminus B$ is the complement of B, $m(B)$ the Lebesgue measure of B and where h is as before the density of $P_1(0, d\mathbf{x})$.

For proving this last lemma, we will need the following L^1-Poincaré, also called $(1,1)$ Poincaré, on balls. This inequality can be in fact though of as a Cheeger type inequality on balls. See [74] and references therein.

Lemma 7.3.8. *For any real $R > 0$, if B denotes the ball centered at 0 and of radius R for the Carnot-Carathéodory distance, there exists a real constant $C > 0$ such that for any smooth $f : \mathbb{H} \to \mathbb{R}$, by denoting $m = \frac{1}{m_B} \int_B f(x) \, d\mathbf{x}$ the mean of f on B,*

$$\int_B |f(x) - m| \, d\mathbf{x} \leq C \int_B |\nabla f| \, (x) \, d\mathbf{x}.$$

We can now make the proof of lemma 7.3.7.

Proof.[proof of lemma 7.3.7] As in lemma 7.3.6, we work with $R = 1$ for simplicity. For any auxillary function $g : \mathbb{H} \to \mathbb{R}$, we have by denoting $m = \int_B f \, d\mathbf{x}$,

$$\int_{B^c} |f - m| \, h \, d\mathbf{x} \leq \int_{B^c} |f - g| \, h \, d\mathbf{x} + \int_{B^c} |g - m| \, h \, d\mathbf{x}.$$

Now we choose g such that $g(\xi, s) = f(\xi, 1)$ where (ξ, s) denotes the polar coordinates in $\mathbb{H}$. More precisely, as we parametrize only the exterior of the ball, we take ξ such that $|\xi| \geq \frac{1}{2\pi}$ and $s \geq 1$ is the Carnot-Carathéodory distance to the origin 0, so that $(\xi, 1)$ is well defined and belongs to the unit sphere of center 0. And so,

$$\begin{aligned} \int_{B^c} |f - m| \, h \, d\mathbf{x} &\leq \int_{|\xi| \geq \frac{1}{2\pi}} \int_{s=1}^{2\pi|\xi|} |f(\xi, s) - f(\xi, 1)| h(\xi, s) A(\xi, s) d\xi ds \\ &+ \int_{|\xi| \geq \frac{1}{2\pi}} \int_{s=1}^{2\pi|\xi|} |f(\xi, 1) - m| h(\xi, s) A(\xi, s) d\xi ds \end{aligned}$$

For the first term the desired gradient bound follows then by the same computation as in lemma 7.3.6. For the second term, we write

$$|f(\xi,1)-m| \leq \int_{u=0}^{1} \left(|f(\xi,1)-f(\xi,u)|+|f(\xi,u)-m|\right) \frac{A(\xi,s)ds}{C(\xi)}$$

where $C(\xi)=\int_{s=0}^{1} A(\xi,s)ds$.
For a clarifying purpose note that the coordinates (ξ,u) for $\xi\frac{1}{2\pi}$ and $0\leq u\leq 1$ do not describe all the unit ball but only a part of it.
By elementary arguments similar as before, one has, using (7.3.29),

$$\begin{aligned}
&\int_{|\xi|\geq\frac{1}{2\pi}} \int_{s=1}^{2\pi|\xi|} \left(\int_{u=0}^{1} |f(\xi,1)-f(\xi,u)|\frac{A(\xi,u)}{C(\xi)}du\right) h(\xi,s)A(\xi,s)d\xi ds\\
\leq& C\int_{|\xi|\geq\frac{1}{2\pi}} \left(\int_{u=0}^{1} |f(\xi,1)-f(\xi,u)|\frac{A(\xi,u)}{C(\xi)}du\right) h(\xi,1)A(\xi,1)d\xi\\
\leq& C\int_{|\xi|\geq\frac{1}{2\pi}} \left(\int_{v=0}^{1} \left(\frac{\int_{u=0}^{v} A(\xi,u)du}{C(\xi)}\right) |\nabla f|(\xi,v)dv\right) h(\xi,1)A(\xi,1)d\xi
\end{aligned}$$

since as before $|f(\xi,1)-f(\xi,u)|\leq\int_{v=u}^{1}|\nabla f|(\xi,v)dv$. Using the above estimates, with $\tau_v=\frac{v}{2|\xi|}$, one also has

$$\frac{A(\xi,v)}{\int_{u=0}^{v} A(\xi,u)du} \simeq \begin{cases} \frac{1}{\tau_v} & \text{if} \quad \tau_v\leq\frac{\pi}{2}\\ \pi-\tau_v & \text{if} \quad \tau_v\geq\frac{\pi}{2}\end{cases}$$

so that for all $v\in(0,1)$,

$$\frac{A(\xi,1)}{\int_{u=0}^{1} A(\xi,u)du} \leq C\frac{A(\xi,v)}{\int_{u=0}^{v} A(\xi,u)du}$$

for some constant C. Noticing that h is bounded above and below on the unit sphere and also on the unit ball ends the proof for this term since we can then bound it above by

$$C\int_{|\xi|\geq\frac{1}{2\pi}} \int_{v=0}^{1} |\nabla f|(\xi,v)dv h(\xi,v)A(\xi,v)d\xi dv.$$

The last term will be bounded with the use of the L^1-Poincaré inequality of lemma 7.3.8.

$$\begin{aligned}
&\int_{|\xi|\geq\frac{1}{2\pi}} \int_{s=1}^{2\pi|\xi|} \left(\int_{u=0}^{1} |f(\xi,u)-m|\frac{A(\xi,u)}{C(\xi)}du\right) h(\xi,s)A(\xi,s)d\xi ds\\
\leq& C\int_{|\xi|\geq\frac{1}{2\pi}} \left(\int_{u=0}^{1} |f(\xi,u)-m|A(\xi,u)du\right) d\xi
\end{aligned}$$

where we use that

$$\int_{s=1}^{2\pi|\xi|} h(\xi,s)A(\xi,s)ds \leq C\int_{0}^{1} A(\xi,u)du.$$

This last inequality is clear in the Euclidean case since both side are constant. Let us check it on the Heisenberg group. For $\tau_1=\frac{1}{2|\xi|}\leq\frac{\pi}{2}$, by the above estimates one has, with $r=|\xi|$,

$$\begin{aligned}
\int_{s=1}^{2\pi|\xi|} h(\xi,s)A(\xi,s)ds &\leq Cr^2\int_{\tau_1}^{\infty} \tau^4 e^{-\tau^2 r^2}d\tau\\
&\leq \frac{C}{r^3}\int_{\frac{1}{2}}^{\infty} y^4 e^{-y^2}dy
\end{aligned}$$

and

$$\begin{aligned}\int_0^1 A(\xi,u)du &\geq Cr^2\int_0^{\tau_1}\tau^4 d\tau\\ &\geq \frac{C}{r^3}.\end{aligned}$$

For $\tau_1 = \frac{1}{2|\xi|} \geq \frac{\pi}{2}$, the inequality is immediate.
Now we can use the L^1-Poincaré inequality of lemma 7.3.8 with noticing as before h is bounded above and below on the unit ball and bound our last term by

$$C\int_B |\nabla f| h dx;$$

which ends the proof. □

Note that lemma 7.3.6 can be deduced directly from lemma 7.3.7. We are now in position to prove the H.-Q. Li inequality (7.0.2).

Proof.[Proof of Theorem 7.3.4] With the help of lemmas 7.3.6 and 7.3.7, we may reduce the study of the H.-Q. Li inequality to functions which are

- either supported in a ball of radius 1 for the Carathéodory metric;
- either supported in a cylinder of radius 2 around the z axis (without the unit ball);
- either supported outside a cylinder around the z-axis.

Indeed, let see how one may reduce first to the case of a function supported either in a ball or outside a ball. If f is any smooth function and ϕ a smooth cutoff function with values 1 on the ball B of radius 1 and vanishing outside a ball of radius 2, we write $f = f\phi + f(1-\phi) = f_1 + f_2$. Clearly, in order to obtain (7.0.2), one can add any prescribed constant to f. In particular, one can assume that $\int_B f\, d\mathbf{x} = 0$. Assuming that we know the inequality for f_1 and f_2, we bound

$$X(P_1 f)(0) = \int \hat{X}(f), h dx \leq C\int (|\nabla f_1| + |\nabla f_2|), h dx$$

then we make use of

$$|\nabla f_1| + |\nabla f_2| \leq |\nabla f| + 2\,|f|\,|\nabla\phi|$$

and since $|\nabla\phi|$ is supported outside the unit ball B,

$$|f|\,|\nabla\phi| \leq ||\nabla\phi||_\infty |f| 1_{B^c}$$

so one has by lemma 7.3.7, recalling $\int_B f d\mathbf{x} = 0$,

$$\int |f| 1_{B^c} h\rangle \leq C\int |\nabla f| h dx$$

and thus

$$\int |f|\,|\nabla\phi|\, h dx \leq C\int |\nabla f|\, h dx.$$

for a different constant C.

We repeat the same operation with a cutoff function for the neighborhood of the z-axis. Let g be a smooth function which vanishes on B. Let ψ be a cut-off function such that $\psi = 1$ on the cylinder of radius 2 and such ψ vanishes on a bigger cylinder. As before we write $g = g\psi + g(1-\psi) = g_1 + g_2$ and we bound, assuming we know the inequality for g_1 and g_2,

$$\begin{aligned}\int \hat{X}(g)hdx &\leq C\int (|\nabla g_1| + |\nabla g_2|)hdx \\ &\leq C\int (|\nabla g| + 2g\|\nabla\psi\|)hdx\end{aligned}$$

and now since g is supported outside the ball B, one may apply lemma 7.3.6 to obtain

$$\int \hat{X}(g)hdx \leq C\int |\nabla g|hdx.$$

Now, when f is supported inside the ball, we may proceed as in the proof of theorem 7.3.1,

$$\begin{aligned}\int \hat{X}(f)hdx &= \int (X + yZ)fhdx \\ &= \int Xfhdx + \int y(XY - YX)fhdx \\ &= \int X(f)\,(yY(\log h) + 2)hdx - \int Y(f)\,yX(\log h)hdx\end{aligned}$$

by an integration by parts. Now, one can conlude using the fact that $|\nabla \log h|\,(\mathbf{x}) \leq Cd(\mathbf{x})$, which is bounded on the unit ball.
If f is supported inside the cylinder around the z-axis and vanishes on the unit ball, we write

$$\int \hat{X}(f)hdx = \int X(f)hdx + \int f\,yZ(\log h)hdx$$

and then we use the fact that $yZ(\log h)$ is bounded on the cylinder. It remains to bound

$$\int |f|\,hdx \leq C\int |\nabla f|\,hdx$$

thanks to lemma 7.3.6.

It remains to deal with a function which is supported outside a cylinder around the z-axis. We shall choose another integration by parts. For that, let us use a complex notation and write

$$\nabla(f) = X(f) + iY(f) \quad \text{and} \quad \hat{\nabla}(f) = \hat{X}(f) - i\hat{Y}(f).$$

Note the change of sign in front of i in the second expression. We want to bound

$$\int \hat{\nabla}(f)hdx = -\int f\hat{\nabla}hdx.$$

Now, since h is radial, we have

$$\hat{\nabla}h = \frac{x - iy}{x + iy}\nabla h$$

which comes from the fact that $\partial_\theta h = 0$ or equivalently$x\partial_y h = y\partial_x h$. Indeed, an easy computation gives

$$(x+iy)\hat{\nabla} = x\partial_x + y\partial_y + iy\partial_x - ix\partial_y + i(x^2+y^2)\partial_z$$

and

$$(x-iy)\nabla = x\partial_x + y\partial_y - iy\partial_x + ix\partial_y + i(x^2+y^2)\partial_z.$$

Let us call $\Psi(x,y)$ the complex function $\frac{x-iy}{x+iy}$, it correponds to the function $\exp(-2i\theta)$ where θ is the angle in the plane (x,y). Then, we integrate again by parts and get

$$\int \hat{\nabla} f h dx = -\int f\Psi(x,y)\nabla h dx = \int \nabla f \Psi(x,y) h dx + \int f, \nabla(\Psi) h dx.$$

We then conclude observing that Ψ is bounded and $|\nabla\Psi|$ is bounded outside the cylinder around the z axis. Indeed, note that, as Ψ does not depend on the z variable, $\nabla\Psi$ is the usual gradient. We therefore have

$$\left|\int \hat{\nabla}(f), h dx\right| \leq \int |\nabla f|\, h dx + C\int |f|\, h dx$$

and we use again lemma 7.3.6 to conclude the proof. $\square$

7.3.3 The H.Q. Li inequality via a complex commutation inequality

In $\mathbb{R}^n$, it is known that the gradient ∇ commute with the Laplace operator. This commutation leads to the commutation between ∇ and the heat semigroup $P_t = e^{t\Delta}$ and therefore to the inequality:

$$\mid \nabla P_t f \mid = \mid P_t \nabla f \mid \leq P_t \mid \nabla f \mid .$$

In the Heisenberg group, we can follow the same pattern of proof. Nevertheless, several difficulties appear that make the proof quite delicate and technical at certain points. For sake of clarity, before we enter the hearth of the proof, let us precise our strategy. The Lie algebra structure:

$$[X,Y] = 2Z, [X,Z] = [Y,Z] = 0$$

leads to the commutation:

$$(X+iY)L = (L-4iZ)(X+iY),$$

where $L = X^2 + Y^2$. At the level of semigroups, it leads to the *formal* commutation:

$$(X+iY)P_t = e^{t(L-4iZ)}(X+iY). \tag{7.3.31}$$

This commutation is only formal because as we will see the semigroup associated to the complex operator $L - 4iZ$ is not globally well defined. More precisely, complex solutions to the heat equation $\frac{\partial u}{\partial t} = (L-4iZ)u$, $u(0,\cdot) = f$ may have poles. Nevertheless, we will see that if the initial condition f is a complex gradient, then solutions to this equation do not explode. In that case, there is moreover an integral representation of the solution. The kernel of this representation being not unique. If we could choose the kernel in such a way that the ratio of it with the density P_t is bounded, then the H.-Q. Li inequality would easily follow. However, we will prove that it is not possible to find such a kernel. To overlap this difficulty, we will use two different kernels depending on the support of the function f. By using a partition of the unity as in our previous proof of H.-Q. Li inequality and a Cheeger type lemma sometimes refered to as the L^1-Poincare inequality on balls (see [74]), we will then be able to conclude.

We now enter into the hearth of the proof. In what follows, in order to exploit the rotational invariance, we shall use the cylindric coordinates $x = r\cos\theta$, $y = r\sin\theta$ in which the vector fields X and Y read

$$X = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta - r\sin\theta\partial_z$$

$$Y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta + r\cos\theta\partial_z$$

$$Z = \partial_z.$$

The heat kernel associated to $(P_t)_{t\geq 0}$ writes here in cylindric coordinates

$$p_t(r,z) = \frac{1}{16\pi^2}\int_{-\infty}^{+\infty} e^{i\lambda\frac{z}{2}} \frac{\lambda}{\sinh\lambda t} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda t} d\lambda. \tag{7.3.32}$$

To give a sense to (7.3.31), we begin with the analytical properties of $p_t(r,z)$ in the variable z.

Lemma 7.3.9. *Let $t > 0$ and $r \geq 0$. The function*

$$z \to p_t(r,z) - \frac{1}{8\pi^2\left(t + i\frac{z}{2} + \frac{r^2}{4}\right)^2} - \frac{1}{8\pi^2\left(t - i\frac{z}{2} + \frac{r^2}{4}\right)^2}$$

admits an analytical extension on $\left\{z \in \mathbb{C}, \mid \mathbf{Im} z \mid < \frac{r^2}{2} + 6t\right\}$. *The function*

$$z \to p_t(r,z)$$

admits therefore a meromorphic continuation in the strip $\left\{z \in \mathbb{C}, \mid \mathbf{Im} z \mid < \frac{r^2}{2} + 6t\right\}$ *with double poles at* $-i\left(2t + \frac{r^2}{2}\right)$ *and* $i\left(2t + \frac{r^2}{2}\right)$.

Proof. Let $t > 0$ and $r \geq 0$. By using the expression (7.3.32) for $p_t(r,z)$, and

$$\frac{1}{\left(t + i\frac{z}{2} + \frac{r^2}{4}\right)^2} = \int_0^{+\infty} e^{-i\lambda\frac{z}{2}} e^{-\lambda t} e^{-\lambda\frac{r^2}{4}} \lambda d\lambda,$$

$$\frac{1}{\left(t - i\frac{z}{2} + \frac{r^2}{4}\right)^2} = \int_0^{+\infty} e^{i\lambda\frac{z}{2}} e^{-\lambda t} e^{-\lambda\frac{r^2}{4}} \lambda d\lambda,$$

we obtain

$$p_t(r,z) - \frac{1}{8\pi^2\left(t + i\frac{z}{2} + \frac{r^2}{4}\right)^2} - \frac{1}{8\pi^2\left(t - i\frac{z}{2} + \frac{r^2}{4}\right)^2}$$
$$= \frac{1}{16\pi^2}\int_{-\infty}^{+\infty} e^{i\lambda\frac{z}{2}} \left(\frac{e^{-\frac{r^2}{4}|\lambda|\mathrm{cotanh}|\lambda|t}}{\sinh \mid\lambda\mid t} - 2e^{-|\lambda|\frac{r^2}{4} - |\lambda|t}\right) \mid\lambda\mid d\lambda$$

and the desired result follows easily. □

For any $t>0$, $r\ge 0$, and $z\in\mathbb{C}-\{-i(t+\frac{1}{8}r^2)\}$ such that $\mid \mathbf{Im} z \mid < \frac{r^2}{2}+6t$, let us denote

$$p_t^*(r,z)=p_t(r,z)-\frac{1}{8\pi^2\left(t+iz+\frac{r^2}{4}\right)^2}.$$

We have the following commutation property.

Proposition 7.3.10. *If $f:\mathbb{H}\to\mathbb{R}$ is a smooth function with compact support, then*

$$(X+iY)P_tf(0)=\int_{\mathbb{H}} p_t^*(r,z+4it)(X+iY)f(r,\theta,z)rdrd\theta dz,\quad t>0.$$

Proof. Due to the identities $[X,Y]=2Z$ and $[X,Z]=[Y,Z]=0$, we have

$$(X+iY)L=(L-4iZ)(X+iY).$$

If $f(r,\theta,z)=e^{i\lambda z}g(r,\theta)$, for some function g, we deduce from the previous commutation,

$$(X+iY)P_tf(0)=e^{4\lambda t}(P_t(X+iY)f)(0)=e^{4\lambda t}\int_{\mathbb{H}} p_t(r,z)((X+iY)f)(r,\theta,z)rdrd\theta dz.$$

Here we used that Z commute with X and Y.
Let us now observe that for $t>0$,

$$(X+iY)\frac{1}{\left(t+i\frac{z}{2}+\frac{r^2}{4}\right)^2}=0$$

and thus

$$(X+iY)p_t^*=(X+iY)p_t.$$

Consequently,

$$(X+iY)P_tf(0)=e^{4\lambda t}\int_{\mathbb{H}} p_t^*(r,z)((X+iY)f)(r,\theta,z)rdrd\theta dz.$$

Now

$$e^{4\lambda t}f(r,\theta,z)=f(r,\theta,z-4it)$$

and the result for the function f follows by integrating by parts with respect to the variable z. For general f, we can conclude by using the Fourier transform of f with respect to the variable z. □

As a first consequence, we deduce that for every $R>0$, there exists a finite constant $C>0$ such that for every smooth function compactly supported inside a Carnot-Carathéodory ball $\mathbf{B}_R$ of radius R,

$$\left|\nabla P_1 f\right|(0)\le CP_1(|\nabla f|)(0).$$

But of course, here, the constant C that we obtain depends on R, and we shall see below that it blows up when $R\to+\infty$.
Now, if $R>0$ is big enough, the ball with radius R contains the region of the Heisenberg group whose cylindric coordinates are of the form $(r=2,\theta\in[0,2\pi],z=0)$ and if f is a smooth function with compact support that vanishes in a ball with radius R, we have the commutation:

$$(X+iY)P_1f(0)=\int_{\mathbb{H}} p_1(r,z+4i)(X+iY)f(r,\theta,z)rdrd\theta dz,\quad t>0.$$

that follows from the fact that $(X+iY)p_t=(X+iY)p_t^*$ and from the fact that the pole of $(r,z)\to p_1(r,z+4i)$ is at $r=2$, $z=0$. The keypoint is then the following estimate:

Proposition 7.3.11. *There exists $R > 0$ such that*

$$\sup_{r^2+|z|\geq R} \frac{| p_1(r, z+4i) |}{p_1(r, z)} < +\infty.$$

Actually, the proof will show that, on the set $\{r^2+ \mid z \mid\geq R\}$, the kernel $p_1(r, z+4i)$ satisfies the same upper bound as the heat kernel $p_1(r, z)$. The proof follows the same lines as the proof of the upper bound for the heat kernel in [19].

Proof. For convenience, and by symmetry, we may assume $z > 0$. Let us first observe that on our domain:

$$p_1(r, z+4i) = \frac{1}{16\pi^2} \int_{-\infty}^{+\infty} e^{-2\lambda} e^{i\lambda \frac{z}{2}} \frac{\lambda}{\sinh \lambda} e^{-\frac{r^2}{4}\lambda \text{cotanh}\lambda} d\lambda \tag{7.3.33}$$

Note also that the functions $\lambda \to \lambda \coth \lambda$ and $\lambda \to \frac{\lambda}{\sinh \lambda}$ are meromorphic on $\mathbb{C}$ with poles in $ik\pi$ for $k \in \mathbb{Z}$, $k \neq 0$. From [19], it is known that for fixed r, z, the function

$$g : \lambda \to -i\lambda \frac{z}{2} + \frac{r^2}{4}\lambda \text{cotanh}\lambda,$$

has a unique critical point in the strip $\{\mid \mathbf{Im}\lambda \mid< \frac{\pi}{2}\}$. This critical point is $i\theta(r, z)$, where $\theta(r, z)$ the unique solution in $(0, \pi)$ of the equation

$$\mu(\theta(r, z))r^2 = 2z, \tag{7.3.34}$$

with $\mu(\theta) = \frac{\theta}{\sin^2\theta} -$ cotan θ. At this critical point, we have

$$g(i\theta(r, z)) = \frac{d^2(r, z)}{4},$$

where $d(r, z)$ is the Carnot-Carathéodory distance from 0 to the point with cylindric coordinates (r, θ, z) (this distance does not depend on θ, that is why it is omitted in the notation). In fact, our function g corresponds to $g(r, z, \lambda) = f(\frac{r}{\sqrt{2}}, \frac{z}{2}, \lambda)$ where f is the function studied in [19].
Moreover the function $s \to \mathbf{Re} g(s + i\theta(r, z))$, grows with $\mid s \mid$, and has a global minimum at $s = 0$, indeed using

$$\coth(u + iv) = \frac{\cosh u \sinh u - i \sin v \cos v}{\sinh^2 u + \sin^2 v}$$

and

$$iv \coth(iv) = v \cot v,$$

a tedious computation shows that

$$\begin{aligned}
&\mathbf{Re}(g(s + i\theta(r, z)) - g(i\theta(r, z))) \\
&= \frac{\sinh^2 2s}{\sinh^2 2s + \sin^2 2\theta(r, z)} (2s \text{ cotanh } 2s - 2\theta(r, z) \text{ cotan } 2\theta(r, z))r^2 \\
&= \frac{\sinh^2 2s}{\sinh^2 2s + \sin^2 2\theta(r, z)} \left((2s \text{ cotanh } 2s - 1) + (1 - 2\theta(r, z) \text{ cotan } 2\theta(r, z))\right) r^2 \\
&\geq 0.
\end{aligned}$$

Let us finally observe that the previous computation also shows that there exists $\delta > 0$ such that for $s \in [-1, 1]$

$$\mathbf{Re} g(s + i\theta(r,z)) \geq \frac{d^2(r,z)}{4} + \delta r^2 s^2.$$

With all this in hands, we can now turn to our proof. We shall proceed in two steps.
Step 1. We show that for any $\eta > 0$,

$$\sup_{r \geq 3, r^2 \geq \eta |z|} \frac{|p_1(r, z+4i)|}{p_1(r,z)} < +\infty.$$

Under this condition on r and z, the critical point $\theta(r,z)$ stays away from the point $i\pi$, the pole in λ of the integrand function. We first start by changing the contour of integration in (7.3.33):

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-2\lambda} e^{i\lambda \frac{z}{2}} \frac{\lambda}{\sinh \lambda} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda} d\lambda \\
=& \int_{\mathbf{Im}\lambda = \theta(\sqrt{r^2-8}, z)} e^{-2\lambda} e^{i\lambda \frac{z}{2}} \frac{\lambda}{\sinh \lambda} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda} d\lambda \\
=& \int_{\mathbf{Im}\lambda = \theta(\sqrt{r^2-8}, z)} e^{i\lambda \frac{z}{2}} \frac{\lambda}{\sinh \lambda} e^{-\left(\frac{r^2}{4}-2\right)\lambda \mathrm{cotanh}\lambda} e^{2\lambda - 2\lambda \mathrm{cotanh}\lambda} d\lambda
\end{aligned}$$

Therefore, by denoting

$$U(\lambda) = e^{2\lambda - 2\lambda \mathrm{cotanh}\lambda} \frac{\lambda}{\sinh \lambda}$$

we get

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} e^{-2\lambda} e^{i\lambda \frac{z}{2}} \frac{\lambda}{\sinh \lambda} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda} d\lambda \right| \\
\leq & \, e^{-\frac{d(\sqrt{r^2-8},z)^2}{4}} \int_{|s|\leq 1} e^{-(r^2-8)\delta^2 s^2} \left| U(s + i\theta(\sqrt{r^2-8}, z)) \right| ds \\
& + e^{-\frac{d(\sqrt{r^2-8},z)^2}{4}} \int_{|s|\geq 1} e^{-(r^2-8)\delta^2} \left| U(s + i\theta(\sqrt{r^2-8}, z)) \right| ds \\
\leq & C_1 \frac{e^{-\frac{d(r,z)^2}{4}}}{r},
\end{aligned}$$

where we used the facts that on the domain on which we work, the difference $d(\sqrt{r^2-8}, z) - d(r,z)$ is uniformly bounded and that the critical point $\theta(\sqrt{r^2-8}, z)$ stays away from the pole $i\pi$ of U. Finally, from the lower estimate of [71], on the considered domain,

$$p_1(r,z) \geq C_2 \frac{e^{-\frac{d(r,z)^2}{4}}}{r}.$$

It concludes the proof of step 1.
Step 2. We show that there exists $\eta > 0$ such that

$$\sup_{|z| \geq 1, r^2 \leq \eta |z|} \frac{|p_1(r, z+4i)|}{p_1(r,z)} < +\infty.$$

In this case, the critical point is close to $i\pi$ and the previous method fails. In [19], the idea was to integrate over a small circle around $i\pi$ and on an horizontal line above the circle. Here we start by giving an analytical representation of

$$p_1(r, z+4i)$$

that is valid on the domain on which we work. As in the previous proof, we assume $z>0$. Due to the Cauchy theorem, we can change the contour of integration in the representation (7.3.32), to get with $0<\varepsilon<\pi$,

$$\begin{aligned}
p_1(r,z) &= \frac{1}{16\pi^2}\sum_{k=1}^{+\infty}\int_{|\lambda-ik\pi|=\varepsilon} e^{i\lambda\frac{z}{2}}\frac{\lambda}{\sinh\lambda}e^{-\frac{r^2}{4}\lambda\mathrm{cotanh}\lambda}d\lambda \\
&= \frac{-i}{16\pi^2}\sum_{k=1}^{+\infty}\int_{|\lambda|=\varepsilon} e^{i(-i\lambda+ik\pi)\frac{z}{2}}\frac{(-i\lambda+ik\pi)}{\sinh(-i\lambda+ik\pi)}e^{-\frac{r^2}{4}(-i\lambda+ik\pi)\mathrm{cotanh}(-i\lambda+ik\pi)}d\lambda \\
&= \frac{1}{16\pi^2}\int_{|\lambda|=\varepsilon} e^{\lambda\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}\left(\sum_{k\geq 1}(-1)^k(-i\lambda+ik\pi)e^{-\pi\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}\right)\frac{d\lambda}{\sin\lambda}
\end{aligned}$$

Now, using for $|x|<1$

$$\sum_{k\geq 1}(-1)^k x^k = \frac{-x}{1+x}$$

and

$$\sum_{k\geq 1}(-1)^k k x^k = \frac{-x}{(1+x)^2}$$

one gets, if ε is such that $\mathrm{cotan}\,\varepsilon \leq \frac{2z}{r^2}$, that the following representation for the heat kernel holds:

$$p_1(r,z) = \frac{-i}{16\pi^2}\int_{|\lambda|=\varepsilon}\frac{e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}}{1+e^{-\pi\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}}\left(\frac{\pi}{1+e^{-\pi\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}}-\lambda\right)\frac{d\lambda}{\sin\lambda}. \tag{7.3.35}$$

Therefore, for $z>0$,

$$\begin{aligned}
&p_1(r,z+4i) \\
=&\frac{-i}{8\pi^2}\int_{|\lambda|=\varepsilon} e^{2i\lambda}\frac{e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}}{1+e^{-\pi\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}}\left(\frac{\pi}{1+e^{-\pi\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}}-\lambda\right)\frac{d\lambda}{\sin\lambda}
\end{aligned}$$

On our domain, if η is small enough, when $r,z\to+\infty$, $\mathbf{Re}(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda)$ goes uniformly on the circle $|\lambda|=\varepsilon$ to $+\infty$. Consequently, on our domain

$$|p_1(r,z+4i)| \leq c_1\left|\int_{|\lambda|=\varepsilon} e^{2i\lambda}e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\,\mathrm{cotan}\,\lambda\right)}(\pi-\lambda)\frac{d\lambda}{\sin\lambda}\right|$$

for some finite positive constant c_1. By choosing $\varepsilon = \pi - \theta(r,z)$, this choice is possible since $\operatorname{cotan} \varepsilon \sim \frac{1}{\varepsilon} \sim \sqrt{\frac{2z}{\pi r^2}}$, we have

$$\int_{|\lambda|=\varepsilon} e^{2i\lambda} e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\operatorname{cotan}\lambda\right)} (\pi-\lambda)\frac{d\lambda}{\sin\lambda} = \int_{|\lambda|=\pi-\theta(r,z)} e^{2i\lambda} e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\operatorname{cotan}\lambda\right)} (\pi-\lambda)\frac{d\lambda}{\sin\lambda},$$

where the function $\theta(r,z)$ has been introduced above. At this stage, we can follow step by step the proof of Theorem 2.17 in [19] (the only difference is in the function V which we take equal to $V(\lambda) = e^{2i\lambda}\frac{\pi-\lambda}{\sin\lambda}$). For the sake of completness, let us do it. We write

$$\int_{|\lambda|=\pi-\theta(r,z)} e^{2i\lambda} e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\operatorname{cotan}\lambda\right)} (\pi-\lambda)\frac{d\lambda}{\sin\lambda} = \int_{|\lambda|=\pi-\theta(r,z)} e^{-f(r,z,\lambda)} V(\lambda)d\lambda$$

with

$$f(r,z,\lambda) = (\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\operatorname{cotan}\lambda\right)$$

and

$$V(\lambda) = e^{2i\lambda}\frac{\pi-\lambda}{\sin\lambda};$$

and we note that f can be decomposed in the way:

$$f = (\pi-\lambda)\frac{z}{2} - \frac{F(r)}{\lambda} + G(\lambda,r)$$

where $F(r) = \frac{\pi r^2}{4}$, G is holomorphic for $|\lambda| < \pi$ and $G(\lambda,r) = O(r^2)$. Let us denote $\lambda_c = \pi - \theta(r,z)$, λ_c is the critical point of f. Therefore:

$$0 = \frac{\partial f}{\partial\lambda}(\lambda_c) = \frac{F}{\lambda_c^2} + G'(\lambda_c) - \frac{z}{2}.$$

It follows that

$$\begin{aligned}
f(\lambda) - f(\lambda_c) &= -\frac{F}{\lambda} + \frac{F}{\lambda_c} + G(\lambda) - G(\lambda_c) - \frac{z}{2}(\lambda-\lambda_c) \\
&= -\frac{F}{\lambda} + \frac{F}{\lambda_c} + \big(G'(\lambda_c) - \frac{z}{2}\big)(\lambda-\lambda_c) + O(r^2(\lambda-\lambda_c)^2) \\
&= -\frac{F}{\lambda} + \frac{F}{\lambda_c} - \frac{F}{\lambda_c^2}(\lambda-\lambda_c) + O(r^2(\lambda-\lambda_c)^2) \\
&= \frac{F}{\lambda_c}\left(2 - \frac{\lambda}{\lambda_c} - \frac{\lambda_c}{\lambda}\right) + O(r^2(\lambda-\lambda_c)^2)
\end{aligned}$$

uniformly for $\varepsilon \le \frac{\pi}{2}$. Now, if we set $\lambda = \varepsilon e^{i\varphi}$, we get

$$f(\lambda) - f(\lambda_c) = \frac{\pi r^2}{\varepsilon}(1-\cos\varphi) + O(r^2\varepsilon^2(1-\cos\phi)). \tag{7.3.36}$$

By the equation for the critical point (7.3.34), the behaviour of ε is given by

$$\varepsilon^2 \sim \frac{\pi r^2}{2z}$$

and so (7.3.36) reads

$$f(\lambda) - f(\lambda_c) \sim \left(\frac{2\varepsilon}{\pi} + O(\varepsilon^4)\right) z(1-\cos\varphi).$$

In particular, for some $\varepsilon_0 > 0$,

$$Re(f(\lambda)) \geq f(\lambda_c) = \frac{d^2(r,z)}{4}$$

if $|\lambda| = \varepsilon \leq \varepsilon_0$. Now, as V admits a simple pole in 0, therefore $V(\lambda) = O(\frac{1}{\varepsilon})$ on the circle $|\lambda| = \varepsilon$. But the circle has length $2\pi\varepsilon$, and therefore,

$$\int_{|\lambda|=\varepsilon} V(\lambda)d\lambda \leq C$$

for some constant C. Thus, one has

$$\int_{|\lambda|=\pi-\theta(r,z)} e^{-f(r,z,\lambda)} V(\lambda)d\lambda \leq Ce^{-\frac{d^2(r,z)}{4}}.$$

In fact, we can improve the estimate if $rd(r,z)$ is big. Indeed, $1-\cos\varphi$ is bounded below by φ^2 so (7.3.36) implies

$$\begin{aligned}\int_{|\lambda|=\pi-\theta(r,z)} e^{-f(r,z,\lambda)} V(\lambda)d\lambda &\leq Ce^{-\frac{d^2(r,z)}{4}} \int_{\varphi=-\pi}^{\pi} e^{\frac{-cr^2\varphi^2}{\varepsilon}} d\varphi \\ &\leq C'e^{-\frac{d^2(r,z)}{4}} \frac{\sqrt{\varepsilon}}{r}.\end{aligned}$$

Finally, recalling on this domain ε^2 behaves like $\frac{r^2}{z}$ and $d(r,z)$ like $\sqrt{z}$, we get the estimate on our domain:

$$\left|\int_{|\lambda|=\pi-\theta(r,z)} e^{2i\lambda} e^{-(\pi-\lambda)\left(\frac{z}{2}-\frac{r^2}{4}\operatorname{cotan}\lambda\right)} (\pi-\lambda)\frac{d\lambda}{\sin\lambda}\right| \leq c_2 \frac{e^{-\frac{d(r,z)^2}{4}}}{\sqrt{rd(r,z)}}$$

for some finite positive constant c_2. Finally, the lower estimate of [71] leads to the conclusion.□

Remark 7.3.12. *In order to extend the H.-Q. to more general situations, it would be interesting to get a proof of the above proposition that would not use the explicit expression for $p_t(r,z)$.*

We can now reprove H.-Q. Li's inequality by using a partition of the unity (which is here simpler than in the previous subsection) and the L^1-Poincaré inequality of lemma 7.3.8 (which was also used in the previous subsection). Let $f : \mathbb{H} \to \mathbb{R}$ be a smooth positive function compactly supported and let $0 \leq \phi \leq 1$ be a smooth function that takes the value 1 on a ball $\mathbf{B}_{R_1}$ and the value 0 outside the ball $\mathbf{B}_{R_2}$ where $R_1 < R_2$, with R_1 big enough. We have

$$\begin{aligned}
(X+iY)P_1 f(0) =&(X+iY)P_1\phi f(0) + (X+iY)P_1(1-\phi)f(0)\\
=&\int_{\mathbb{H}} p_1^*(r,z+4i)(X+iY)(f\phi)(r,\theta,z)rdrd\theta dz\\
&+\int_{\mathbb{H}} p_1(r,z+4i)(X+iY)(f(1-\phi))(r,\theta,z)rdrd\theta dz\\
=&\int_{\mathbb{H}} \phi(r,\theta,z)p_1^*(r,z+4i)(X+iY)f(r,\theta,z)rdrd\theta dz\\
&+\int_{\mathbb{H}} (1-\phi(r,\theta,z))p_1(r,z+4i)(X+iY)f(r,\theta,z)rdrd\theta dz\\
&+\frac{1}{8\pi^2}\int_{\mathbb{H}} f(r,\theta,z)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+i\frac{z}{2}+\frac{r^2}{4}\right)^2}rdrd\theta dz.
\end{aligned}$$

Therefore

$$\mid \nabla P_1 f(0) \mid \le CP_1 \mid \nabla f \mid (0) + \left| \frac{1}{8\pi^2}\int_{\mathbb{H}} f(r,\theta,z)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+i\frac{z}{2}+\frac{r^2}{4}\right)^2}rdrd\theta dz \right|.$$

Now, we estimate $\left|\int_{\mathbb{H}} f(r,\theta,z)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+i\frac{z}{2}+\frac{r^2}{4}\right)^2}rdrd\theta dz\right|$ thanks to lemma 7.3.8:

$$\begin{aligned}
&\left|\int_{\mathbb{H}} f(r,\theta,z)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+i\frac{z}{2}+\frac{r^2}{4}\right)^2}rdrd\theta dz\right|\\
=&\left|\int_{\mathbb{H}} (f(r,\theta,z)-m)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+i\frac{z}{2}+\frac{r^2}{4}\right)^2}rdrd\theta dz\right| \qquad (m \text{ is the mean of } f \text{ on } \mathbf{B}_{R_2})\\
\le& C_1\int_{\mathbf{B}_{R_2}} \mid f(r,\theta,z)-m \mid rdrd\theta dz\\
\le& C_2\int_{\mathbf{B}_{R_2}} \mid \nabla f \mid (r,\theta,z)rdrd\theta dz\\
\le& C_3 P_1 \mid \nabla f \mid (0).
\end{aligned}$$

This completes the proof of H.-Q. Li's inequality. As we mentioned it in the beginning of this section, interestingly, it is not possible to find a function ϕ on $\mathbb{H}$ such that:

- $(X+iY)\phi = 0$;
- The ratio $\frac{\left|p_1^*(r,z+4i)-\Phi(r,\theta,z)\right|}{p_1(r,z)}$ is bounded.

Indeed, as

$$(X+iY)(r^2+iz) = (X+iY)(re^{i\theta}) = 0,$$

the first point implies that Φ can be written:

$$\Phi(r,\theta,z) = H\left(\frac{r^2}{4} + i\frac{z}{2}, re^{i\theta}\right),$$

where $H : \{z_1 \in \mathbb{C}, \mathbf{Re}(z_1) \geq 0\} \times \mathbb{C} \to \mathbb{C}$ is analytic in z_1 and z_2. Now, due to the estimate of Proposition 7.3.11 and the estimate on p_1, it would imply that for r and z, such that $r^2 + \mid z \mid$ is big enough:

$$\left| H\left(\frac{r^2}{4} + i\frac{z}{2}, re^{i\theta}\right) + \frac{1}{8\pi^2\left(-1 + i\frac{z}{2} + \frac{r^2}{4}\right)^2} \right| \leq Ae^{-B(r^2+|z|)}$$

where A and B are strictly positive constants. Now, we have the following lemma that prevents the the existence of such H:

Lemma 7.3.13. *Let $f : \{z_1 \in \mathbb{C}, \mathbf{Re}(z_1) \geq 0\} \times \mathbb{C} \to \mathbb{C}$ be analytic in z_1 and z_2. If there exist strictly positive constants A and B such that*

$$\forall r \geq 0,\ \forall z \in \mathbb{R},\ \forall \theta \in [0, 2\pi], \quad \left| f\left(r^2 + iz, re^{i\theta}\right) \right| \leq Ae^{-B(r^2+|z|)}$$

then $f = 0$.

Proof.
Let $r \geq 0, z \in \mathbb{R}$. The function $z_2 \to f\left(r^2 + iz, z_2\right)$ is analytic, therefore from the maximum principle we have

$$\begin{aligned} \left| f\left(r^2 + iz, z_2\right) \right| &\leq \max_{\theta\in[0,2\pi]} f(r^2 + iz, re^{i\theta}) \\ &\leq Ae^{-(B|r|^2+|z|)} \\ &\leq Ae^{-(B|z_2|^2+|z|)} \end{aligned}$$

for $\mid z_2 \mid \leq r$. Consequently, on the set $\mathbf{Re}(z_1) \geq \mid z_2 \mid^2$ we have

$$|f(z_1, z_2)| \leq Ae^{-B(|z_2|^2+|\mathbf{Im}(z_1)|)}.$$

By setting $g : z_1 \to e^{-z_1} f(z_1, 0)$, we obtain a function g analytic on the set $\mathbf{Re}(z) > 0$ such that

$$|g(z)| \leq \alpha e^{-\beta|z|}$$

with $\alpha, \beta > 0$, and such function has to be 0. Indeed, by composing it with the homography $\omega \mapsto \frac{1+\omega}{1-\omega}$ which sends the unit disc D to half space space $Re(Z) > 0$, we obtain a function h holomorphic on D and such:

$$|h(\omega)| = |g\left(\frac{1+\omega}{1-\omega}\right)| \leq A\exp\left(-B\left|\frac{1+\omega}{1-\omega}\right|\right).$$

Now, let $\omega \in D$ and $|\omega| < r < 1$, as the function $\log|h|$ is subharmonic, we get

$$\log(|h(\omega)|) \leq \log A + \frac{1}{2\pi}\int_0^{\pi} -B\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right| d\theta.$$

Then, as by letting $r \to 1$,

$$\int_0^\pi \left| \frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right| d\theta \to +\infty$$

and

$$\log(|h(\omega)|) = 0$$

which means $h(\omega) = 0$ and of course implies $g \equiv 0$. This elegant proof was communicated to us by Pascal Thomas. $\square$

7.3.4 The complex commutation on $\mathbb{H}_n$

The generalisation of the H.Q. Li inequality to the isotropic Heisenberg group $\mathbb{H}_n$ of dimension $2n+1$ was done by Elredge [41]. In a first paper [40] (see also [71]), he obtains the optimal estimates of the heat kernel on $\mathbb{H}_n$, then his method is an extension of our method via the Cheeger inequality. However in this section we will see some interesting features of the complex commutation between the semigroup and the gradient on the isotropic Heisenberg group $\mathbb{H}_n$ of dimension $2n+1$ which may also lead to another proof of the H.Q. Li inequality. For what we are interested in here, we only need to know that $\mathbb{H}_n$ is a Lie group, isomorphic to $\mathbb{R}^{2n+1}$ as a manifold, and that there exists a basis of the Lie given by the matrices $\{(X_j, Y_j\}_{1 \le j \le n}$ and Z such that the only non-vanishing Lie brackets are:

$$[X_j, Y_j] = 2Z.$$

The sublaplacian is then given by $L = \sum_{j=1}^n X_j^2 + Y_j^2$ where $\{(X_j, Y_j\}_{1 \le j \le n}$ denote the left invariant vector fields generated by the corresponding matrices and the $2n+1$-dimensional Lebesgue measure is an invariant measure for L and the heat kernel associated to L and issued from the identity writes against this Lebesgue measure as:

$$h_t^n(r,z) = \frac{1}{(4\pi t)^{n+1}} \int_{-\infty}^{+\infty} e^{\frac{i\lambda z}{2t}} e^{-\frac{r^2}{4t}\lambda \coth\lambda} \left(\frac{\lambda}{\sinh \lambda} \right)^n d\lambda. \tag{7.3.37}$$

where $r^2 = \sum_{j=1}^n x_j^2 + y_j^2$. For more details on all these properties, one can coonsult [19].
Now for all dimension n and $1 \le j \le n$, we have the following commutation which do not depend on the dimension:

$$(X_j + iY_j)L = (L - 4iZ)(X_j + iY_j).$$

Let us take a look at $h_t^n(r, z + 4it)$, it reads:

$$h_t^n(r, z+4it) = \frac{1}{(4\pi t)^{n+1}} \int_{-\infty}^{+\infty} e^{-2\lambda} e^{\frac{i\lambda z}{2t}} e^{-\frac{r^2}{4t}\lambda \coth\lambda} \left(\frac{\lambda}{\sinh \lambda} \right)^n d\lambda.$$

This integral defining $h_t^n(r, \tilde{z})$ is absolutely converging as soon as:

$$\frac{|Im\tilde{z}|}{2t} - n - \frac{r^2}{4t} < 0,$$

that is

$$|Im\tilde{z}| < \frac{r^2}{2} + 2nt.$$

Therefore when $n \ge 3$, the kernel $h_t^n(r, z+4it)$ is well defined. It do not admit poles. Note for $n = 2$, it must have a pole only when $r = 0$ and $z = 0$.
As a consequence, when $n \le 3$, we can establish the following representation result.

Proposition 7.3.14. *Let $n \geq 3$, if $f : \mathbb{H}_n \to \mathbb{R}$ is a smooth function with compact support, then*

$$(X_j + iY_j)P_t f(0) = \int_{\mathbb{H}} h_t^n(r, z + 4it)(X_j + iY_j)f(r, \theta, z) r dr d\theta dz, \quad t > 0.$$

Remark 7.3.15. *We now conjecture that the ratio $\frac{|h_t^n(r,z+4it)|}{h_t^n(r,z)}$. With this, we will recover directly the H.Q. Li inequality for $\mathbb{H}_n$ when $n \geq 3$.*

7.3.5 The consequences on $\mathbb{H}$

In this section, we just collect the best functional inequalities available of Section 7.2 on the Heisenberg group using the optimal constants C_1 and C_2 of Theorems 7.3.4 and 7.3.1 and the optimal reverse Poincaré inequality (Proposition 6.2.4).

Proposition 7.3.16. *Let f a smooth function on $\mathbb{H}$ with compact support and $t > 0$, we have:*

- *H.Q. Li inequality*
$$\sqrt{\Gamma(P_t f)} \leq C_1 P_t(\sqrt{\Gamma f}); \tag{7.3.38}$$

- *Driver-Melcher inequality*
$$\Gamma(P_t f) \leq C_2 P_t(\Gamma f); \tag{7.3.39}$$

- *Poincaré inequality,*
$$P_t(f^2) - (P_t(f))^2 \leq 2\, C_2\, t\, P_t(|\nabla f|^2); \tag{7.3.40}$$

- *Beckner-Latała-Oleszkiewicz type inequality*
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \leq p\, C_1^2\, t\, P_t(f^{p-2}|\nabla f|^2); \tag{7.3.41}$$

- *logarithmic Sobolev inequality*
$$P_t(f \log(f)) - P_t(f) \log(P_t(f)) \leq C_1^2\, t\, P_t\left(f^{-1}\, |\nabla f|^2\right); \tag{7.3.42}$$

- *Reverse Poincaré inequality,*
$$P_t(f^2) - (P_t(f))^2 \geq t\, P_t(\Gamma(f)); \tag{7.3.43}$$

- *Beckner-Latała-Oleszkiewicz type inequality*
$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \geq p\, \frac{t}{C_1^2}(P_t f)^{p-2}\Gamma(P_t f); \tag{7.3.44}$$

- *Reverse logarithmic Sobolev inequality*
$$P_t(f \log(f)) - P_t(f) \log(P_t(f)) \geq \frac{t}{C_1^2}\, \frac{\Gamma(P_t f)}{P_t f}; \tag{7.3.45}$$

- *A uniform gradient bound:*

$$\|\sqrt{\Gamma P_t f}\|_\infty \leq \frac{1}{\sqrt{t}}\|f\|_\infty; \tag{7.3.46}$$

- *Cheeger type inequality*

$$P_t(|f - P_t(f)(\mathbf{x})|)(\mathbf{x}) \leq 4C_1\sqrt{t}\, P_t(\sqrt{\Gamma f})(\mathbf{x}). \tag{7.3.47}$$

- *A first Bobkov type isoperimetric inequality*

$$\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) \leq C_1^2\sqrt{2t}\, P_t(\sqrt{\Gamma f}). \tag{7.3.48}$$

- *Another Bobkov type isopermetric inequality*

$$\mathcal{I}(P_t f) \leq P_t\left(\sqrt{(\mathcal{I}(f))^2 + 2C_1^4 t\,\Gamma(f)}\right) \tag{7.3.49}$$

- *A reverse Bobkov type isoperimetric inequality*

$$\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) \geq \sqrt{2t}C_1^2\sqrt{\Gamma(P_t f)}. \tag{7.3.50}$$

7.4 The cases of $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$

For the moment, it does not seem possible to obtain an inequality of the type of (7.0.2) by using the above methods since both of the them rely on optimal estimates of the heat kernel which are not known yet. The method via the Cheeger inequality uses expression of the geodesics and the Jacobian in well-chosen geodesic coordinates. The things are available on both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$, the computations of the geodesics are done in [26] and the computation of the Jacobian in some geodesic coordinates in [31] but, as we just said, the optimal estimates of the heat kernel are not known.

For the method via the complex commutation, we can not of course obtain the full inequality (7.0.2) (because of the lack of the optimal estimates for the heat kernel) but we can see that some of the main features of this complex commutation are shared by $\mathbb{H}$, $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$.

7.4.1 The complex commutation

Here we work with the Lie algebra with the parameter ρ introduced in Chapter 1. Then the Lie algebra relations lead to the general commutation:

$$(X + iY)L = (L - 4iZ + 4\rho)(X + iY). \tag{7.4.51}$$

Indeed, one has:

$$\begin{aligned}
[X, L] &= [X, Y^2] \\
&= [X, Y]Y + Y[X, Y] \\
&= 2ZY + 2YZ \\
&= 2[Y, Z] + 4ZY \\
&= 4\rho X + 4ZY.
\end{aligned}$$

and

$$\begin{aligned}
[Y,L] &= [Y,X^2] \\
&= [Y,X]X + X[Y,X] \\
&= -2ZX - 2XZ \\
&= -2[X,Z] - 4ZX \\
&= 4\rho Y - 4ZX.
\end{aligned}$$

Remark 7.4.1. *Note that in these computations, we do not arrange the terms in the same way as in the computation of the Γ_2 (see 2.1.7). The goal in the computation of the Γ_2 was indeed to make appear the term in Γ to obtain some curvature-dimensio type inequality whereas here it is to obtain a commutation with the complex gradient $(X+iY)$.*

Then equality 7.4.51 leads to the formal commutation

$$(X+iY)P_t = e^{t(L-4iZ+4\rho)}(X+iY). \tag{7.4.52}$$

In what follows we give a precise analytical sense to the previous commutation for both $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$ by looking at the analytical properties of the heat kernel $p_t(r,z)$ in the z variable. The case of the Heisenberg group was done in the previous section.
First we treat the case of the $\mathbf{SU}(2)$ group that is $\rho = 1$.

Lemma 7.4.2. *Let $t>0$ and $r\ge 0$. The function*

$$z \to p_t(r,z) - \frac{1}{2\pi^2}\frac{1}{(1-\cos r e^{iz-2t})^2} - \frac{1}{2\pi^2}\frac{1}{(1-\cos r e^{-iz-2t})^2}$$

admits an analytical continuation on $\{z\in\mathbb{C}, \mid \mathbf{Im} z \mid < -\ln\cos r + 6t\}$. The function

$$z \to p_t(r,z)$$

is therefore meromorphic in the strip $\{z\in\mathbb{C}, \mid \mathbf{Im} z \mid < -\ln\cos r + 6t\}$ with double poles at $-i\,(-\ln\cos r + 2t)$ and $i\,(-\ln\cos r + 2t)$.

Proof. This is an easy consequence of the spectral decomposition of p_t:

$$p_t(r,z) = \sum_{n=-\infty}^{+\infty}\sum_{k=0}^{+\infty}(2k+\mid n\mid+1)e^{-(4k(k+|n|+1)+2|n|)t}e^{inz}(\cos r)^{|n|}P_k^{0,|n|}(\cos 2r).$$

Indeed, using for $|x|\le 1$,

$$\frac{1}{(1-x)^2} = \sum_{n\ge 0}(n+1)x^n,$$

one has

$$\frac{1}{(1-\cos r e^{-iz-2t})^2} = \sum_{n\ge 0}(n+1)(\cos r)^n e^{-inz}e^{-2nt}$$

and

$$\frac{1}{(1-\cos r e^{iz-2t})^2} = \sum_{n\ge 0}(n+1)(\cos r)^n e^{inz}e^{-2nt}.$$

Therefore,

$$2\pi^2 p_t(r,z) - \frac{1}{(1-\cos r e^{iz-2t})^2} - \frac{1}{(1-\cos r e^{-iz-2t})^2}$$
$$= \left(\sum_{n=-\infty}^{+\infty} \sum_{k=1}^{+\infty} (2k+\mid n \mid +1) e^{-(4k(k+|n|+1)+2|n|)t} e^{inz} (\cos r)^{|n|} P_k^{0,|n|}(\cos 2r) \right) - 1$$

and this serie is absolutely convergent as soon as both $|e^{iz}\cos r| < 1$ and $|e^{-iz}\cos r| < 1$; which gives the desired result. $\square$

Let us now observe that if $k = 0$ and $n \leq 0$,

$$(X+iY)e^{inz}(\cos r)^{|n|} P_k^{0,|n|}(\cos 2r) = 0.$$

If, for $t > 0$, $r \geq 0$, $z \in \mathbb{C} - \{-i(-\ln \cos r + 2t)\}$, $\mid \mathbf{Im} z \mid < -\ln \cos r + 6t$, we denote

$$p_t^*(r,z) = p_t(r,z) - \frac{1}{2\pi^2}\frac{1}{(1-\cos r e^{-iz-2t})^2},$$

we have therefore

$$(X+iY)p_t = (X+iY)p_t^*.$$

Note that, using some previous computations, $p_t^*(r,z)$ writes

$$\begin{aligned} 2\pi^2 p_t^*(r,z) &= \sum_{n=-\infty}^{+\infty} \sum_{k=1}^{+\infty} (2k+\mid n \mid +1) e^{-(4k(k+|n|+1)+2|n|)t} e^{inz} (\cos r)^{|n|} P_k^{0,|n|}(\cos 2r) \\ &+ \sum_{n\geq 1} (n+1) e^{inz} (\cos r)^n e^{-2nt}. \end{aligned}$$

Combining this with (7.4.52) leads to:

Proposition 7.4.3. *If $f : \mathbf{SU}(2) \to \mathbb{R}$ is a smooth function, then*

$$(X+iY)P_t f(0) = e^{4t} \int_{\mathbb{H}} p_t^*(r, z+4it)(X+iY)f(r,\theta,z)d\mu, \quad t > 0.$$

Proof. Indeed, by the commutation (7.4.52), one has

$$\begin{aligned} (X+iY)P_t(f)(0) &= e^{4\rho t} P_t(e^{-4itZ}(X+iY)f)(0) \\ &= e^{4\rho t} \int e^{-4itZ}(X+iY)f(r,\theta,z)\, p_t(r,z) d\mu(r,\theta,z) \\ &= e^{4\rho t} \int e^{-8\rho t}(X+iY)(e^{-4itZ}f)(r,\theta,z)\, p_t(r,z) d\mu(r,\theta,z) \\ &= e^{4\rho t} \int e^{-8\rho t}(X+iY)(e^{-4itZ}f)(r,\theta,z)\, p_t^*(r,z) d\mu(r,\theta,z) \\ &= e^{4\rho t} \int e^{-4itZ}(X+iY)(f)(r,\theta,z)\, p_t^*(r,z) d\mu(r,\theta,z) \\ &= e^{4\rho t} \int (X+iY)(f)(r,\theta,z)\, e^{4itZ} p_t^*(r,z) d\mu(r,\theta,z) \\ &= e^{4\rho t} \int (X+iY)(f)(r,\theta,z)\, p_t^*(r,z+4it) d\mu(r,\theta,z) \end{aligned}$$

since $z \to p_t^*(r,z)$ is holomorphic for $|Imz| \leq 6t + \frac{r^2}{2}$. □

Now we turn to the $\mathbf{SL}(2,\mathbb{R})$ group that is $\rho = -1$.

Lemma 7.4.4. *Let $t > 0$ and $r \geq 0$. The function*

$$z \to p_t(r,z) - \frac{1}{2\pi^2}\frac{1}{\left(1 - \frac{1}{\cosh r}e^{iz-2t}\right)^2} - \frac{1}{2\pi^2}\frac{1}{\left(1 - \frac{1}{\cosh r}re^{-iz-2t}\right)^2}$$

admits an analytical continuation on $\{z \in \mathbb{C}, \mid \mathbf{Im}z \mid < \ln\cosh r + 6t\}$. The function

$$z \to p_t(r,z)$$

is therefore meromorphic in the strip $\{z \in \mathbb{C}, \mid \mathbf{Im}z \mid < \ln\cosh r + 6t\}$ with double poles at $-i\,(\ln\cosh r + 2t)$ and $i\,(\ln\cosh r + 2t)$.

Proof.
When $|y|$ tends to infinity, we have

$$\mathrm{arch}(\cosh r \cosh y) = |y| + \ln(\cosh r) + O(e^{-2|y|})$$

so

$$\mathrm{arch}(\cosh r\cosh y)^2 - y^2 = 2|y|\ln(\cosh r) + \ln(\cosh r)^2 + O(|y|e^{-2|y|}).$$

Also

$$\frac{1}{\sqrt{\cosh^2 r\cosh^2 y - 1}} = \frac{2e^{-|y|}}{\cosh r}(1 + O(e^{-2|y|}))$$

and then

$$e^{-\frac{z^2}{4t}}e^{-\frac{\mathrm{arch}^2(\cosh r\cosh y) - y^2}{4t}}\frac{\mathrm{arch}(\cosh r\cosh y)}{\sqrt{\cosh^2 r\cosh^2 y - 1}}$$

$$= \exp\left(\frac{-|y|\ln(\cosh r)}{2t}\right)\exp(-|y|)\exp\left(-\frac{z^2}{4t} - \frac{\ln(\cosh r)^2}{4t} - \ln(\cosh r)\right)(1 + O(|y|e^{-2|y|})).$$

So eventually p_t is equal to:

$$p_t(r,z) = 2\frac{e^{-t}}{(4\pi t)^2}e^{\left(-\frac{z^2}{4t} - \ln(\cosh r) - \frac{\ln(\cosh r)^2}{4t}\right)}\int_{-\infty}^{+\infty} e^{\frac{-iyz}{2t}}e^{\frac{-|y|\ln(\cosh r)}{2t}}e^{-|y|}\left(|y| + \ln(\cosh r) + O(|y|e^{-2|y|})\right)dy.$$

Now we can see that

$$\frac{1}{4t^2}\int_{-\infty}^{+\infty} e^{\frac{-iyz}{2t}}e^{\frac{-|y|\ln(\cosh r)}{2t}}e^{-|y|}\left(|y| + \ln(\cosh r)\right)dy$$

$$= \frac{1}{(2t + \ln(\cosh r) + iz)^2} + \frac{1}{(2t + \ln(\cosh r) - iz)^2} + \frac{\ln(\cosh r)}{2t(2t + \ln(\cosh r) + iz)} + \frac{\ln(\cosh r)}{2t(2t + \ln(\cosh r) - iz)}.$$

To finish the proof we only have to check, by calculating the beginning of the Laurent series of each term, that

$$\frac{1}{\left(1 - \frac{1}{\cosh r}e^{iz-2t}\right)^2} - \left(\frac{1}{(2t + \ln(\cosh r) + iz)^2} + \frac{\ln(\cosh r)}{2t(2t + \ln(\cosh r) + iz)}\right)e^{\left(-\frac{z^2}{4t} - \ln(\cosh r) - \frac{\ln(\cosh r)^2}{4t}\right)}e^{-t}$$

and

$$\frac{1}{\left(1-\frac{1}{\cosh r}e^{-iz-2t}\right)^2}-\left(\frac{1}{(2t+\ln(\cosh r)-iz)^2}+\frac{\ln(\cosh r)}{2t(2t+\ln(\cosh r)-iz)}\right)e^{\left(-\frac{z^2}{4t}-\ln(\cosh r)-\frac{\ln(\cosh r)^2}{4t}\right)}e^{-t}$$

are holomorphic functions. □

Let us now observe that

$$(X+iY)(\ln\cosh r+iz)=0$$

so that

$$(X+iY)\frac{1}{\left(1-\frac{1}{\cosh r}e^{-iz-2t}\right)^2}=0.$$

If, for $t>0$, $r\geq 0$, $z\in\mathbb{C}-\{-i\,(\ln\cosh r+2t)\}$, $\mid \mathbf{Im}z\mid<\ln\cosh r+6t$, we denote

$$p_t^*(r,z)=p_t(r,z)-\frac{1}{2\pi^2}\frac{1}{\left(1-\frac{1}{\cosh r}e^{-iz-2t}\right)^2},$$

we have therefore

$$(X+iY)p_t=(X+iY)p_t^*.$$

Combining this with (7.4.52) leads to:

Proposition 7.4.5. *If $f:\mathbf{SL}(2,\mathbb{R})\to\mathbb{R}$ is a smooth function, then*

$$(X+iY)P_tf(0)=e^{-4t}\int_{\mathbb{H}}p_t^*(r,z+4it)(X+iY)f(r,\theta,z)d\mu,\quad t>0.$$

Proof. The proof is the same than on $\mathbf{SU}(2)$. □

7.4.2 The exponential decay in big times on $\mathbf{SU}(2)$

As a corollary of the proposition 7.4.3 about the commutation between a complex gradient and the semigroup, one gets

Corollary 7.4.6. *There exists $t_0>0$ and $A>0$ such that for any smooth $f:\mathbf{SU}(2)\to\mathbb{R}$,*

$$\sqrt{\Gamma(P_tf,P_tf)(0)}\leq Ae^{-2t}P_t\sqrt{\Gamma(f,f)}(0),\quad t\geq t_0.$$

Remark 7.4.7. *We can observe that the constant that appears in the commutation in the proposition 7.4.3 is positive, which is quite striking because we expect an exponential decay. Nevertheless, as we will see below $e^{t(L-4iZ)}$ gives a decay e^{-6t} against complex gradients.*

Proof. We denote

$$\Phi(t)=\sup_{r\in[0,\frac{\pi}{2}]}\sup_{z\in[-\pi,\pi]}\frac{\mid p_t^*(r,z+4it)\mid}{p_t(r,z)}.$$

Since,

$$\begin{aligned}p_t^*(r,z+4it)=&\sum_{n=1}^{+\infty}(n+1)e^{-6nt}e^{inz}(\cos r)^n\\&+\sum_{n=-\infty}^{+\infty}\sum_{k=1}^{+\infty}(2k+\mid n\mid+1)e^{-(4k(k+|n|+1)+2|n|)t}e^{inz}e^{-4nt}(\cos r)^{|n|}P_k^{0,|n|}(\cos 2r)\end{aligned}$$

there exists $t_0 > 0$ and $A > 0$, such that for $t \geq t_0$,

$$\Phi(t) \leq \frac{\mid p_t^*(0) \mid}{1- \mid 1 - p_t(0) \mid} \leq Ae^{-6t},$$

and the proof is complete. □

Remark 7.4.8. *The above function $\Phi(t)$ explodes when $t \to 0$.*

Remark 7.4.9. *We conjecture that the ratio $\sup_{r,z \notin \Sigma_{R\sqrt{t}}} \frac{p_t(r,z+4it)}{p_t(r,z)}$ is bounded when $t \to 0$, where R is big enough and $\Sigma_{R\sqrt{t}}$ denotes the Carnot Carathéodory ball with radius $R\sqrt{t}$. By a partition of unity similar to the one done in the Heisenberg case, this would imply that Corollary 7.4.6 also holds for all $t > 0$.*

Remark 7.4.10. *As mentionned in [70], this result can also be obtained by a rather elementary way in a wider setting. To see this, we will work for simplicity on $\mathcal{M}$ a compact Riemannian manifold without boundary. Of course this is also true in our setting of $\mathbf{SU}(2)$. We consider the semigroup associated to the Laplace-beltrami operator. We call μ an invariant measure, $m = \mu(M)$ its total mass and $\lambda_1 > 0$ its first eigenvalue. Then for all smooth function f on M and all $x \in M$,*

$$\begin{aligned} |\nabla(P_t f)(x)| &= |\nabla_x \int_M p_t(x,x') \left(f(x') - \frac{1}{m} \int_M f(x_*) d\mu(x_*) \right) | \\ &\leq \int_M |\nabla_x p_t(x,x')| \, . \, |f(x') - \frac{1}{m} \int_M f(x_*) d\mu(x_*)| d\mu(x') \end{aligned}$$

But, $|\nabla_x p_t(x,x')|$ is in this setting bounded above by $C_1 e^{-\lambda_1 t}$ uniformly for $x, x' \in M$ and t big enough. Now, by the Poincaré inequality for the invariant measure μ

$$\int_M |f(x') - \frac{1}{m} \int_M f(x_*) d\mu(x_*)| d\mu(x') \leq C_2 \, diam(M) \int_M |\nabla f|(x') d\mu(x').$$

Thus,

$$|\nabla(P_t f)(x)| \leq C_3 e^{-\lambda_1 t} \, diam(M) \int_M |\nabla f|(x') d\mu(x')$$

Noticing that, unifomly on $x, x' \in M$, $p_t(x,x') \geq \frac{1}{2m}$ for t big enough, one obtains that

$$\begin{aligned} |\nabla(P_t f)(x)| &\leq C_3 \, m \, diam(M) \int_M |\nabla f|(x') p_t(x,x') d\mu(x') \\ &= C_3 \, m \, diam(M) \, P_t(|\nabla f|)(x). \end{aligned}$$

This exponential decreasing $e^{-\lambda_1 t}$ is better in big times than the one obtained via the $CD(\rho,\infty)$ criterion: $e^{-\max(0,\rho)t}$ for Riemannain manifolds. Recall from the Lichnerowicz theorem that

$$\lambda_1 \geq \frac{n}{n-1} \rho$$

where $n \geq 2$ is the dimension of the manifold M.
On $\mathbf{SU}(2)$, this method works and gives the same decay since $\lambda_1 = 2$.

7.4.3 The behavior in small times on $\mathbf{SU}(2)$

Theorem 7.4.11. *Let $p > 1$. There exists a constant $C_p > 1$ such that for any smooth $f : \mathbf{SU}(2) \to \mathbb{R}$ and any $g \in \mathbf{SU}(2)$*

$$\sqrt{\Gamma(P_t f, P_t f)(g)} \le C_p e^{-2t} \left(P_t \Gamma(f,f)^{\frac{p}{2}}(g) \right)^{\frac{1}{p}}, \quad t \ge 0.$$

Remark 7.4.12. *Let $f(r,\theta,z) = \cos r \cos z$. In that case, $\mathcal{L} f = -2f$ and $\Gamma(f,f) = \sin^2 r$. Therefore, as noticed before, the exponential decay e^{-2t} is optimal and moreover:*

$$\sin r \le C_p P_t (\sin r)^p$$

which implies, by letting $t \to \infty$, $C_p \ge \left(1 + \frac{p}{2}\right)^{\frac{1}{p}}$.

To prove Theorem 7.4.11, we only have to show that the inequality does not explode when $t \to 0$. We shall use here the commutation between left-invariant and right invariant vector fields. It relies on the following lemma:

Lemma 7.4.13. *Let $q > 1$. The limit*

$$\lim_{t \to 0} \int_{\mathbf{SU}(2)} (\sin 2r)^q \Gamma(\ln p_t, \ln p_t)^{\frac{q}{2}}(r,z) p_t(r,z) d\mu$$

is finite.

Proof. The proof is similar to the proof of Proposition 6.2.7: By scaling and a dominated convergence argument based on Proposition 5.2.13, we obtain:

$$\begin{aligned} &\lim_{t \to 0} \int_{\mathbf{SU}(2)} (\sin 2r)^q \Gamma(\ln p_t, \ln p_t)^{\frac{q}{2}}(r,z) p_t(r,z) d\mu \\ =&2^q \int_{\mathbb{R}^3} r^q h_1(r,z) \left((\tilde{X} \ln h_1)^2(r,z) + (\tilde{Y} \ln h_1)^2(r,z) \right)^{q/2} r dr d\theta dz, \end{aligned}$$

which is finite, due to known results on the Heisenberg group. □

We can now deduce:

Proposition 7.4.14. *Let $p > 1$. There exists a constant $A_p > 0$ such that for any smooth $f : \mathbf{SU}(2) \to \mathbb{R}$ and any $g \in \mathbf{SU}(2)$*

$$\sqrt{\Gamma(P_t f, P_t f)(g)} \le A_p \left(P_t \Gamma(f,f)^{\frac{p}{2}}(g) \right)^{\frac{1}{p}}, t \in (0,1).$$

Proof. Due to the fact that the right-invariant vector fields $\hat{X}, \hat{Y}$ commute with $\mathcal{L}$, we get

$$(X P_t f)(0) = (P_t \hat{X} f)(0)$$

and

$$(Y P_t f)(0) = (P_t \hat{Y} f)(0).$$

Now, X, Y, Z form a basis at each point, there exist therefore smooth functions such that:

$$\hat{X} = \Omega_{1,1} X + \Omega_{1,2} Y + \Omega_{1,3} Z$$

$$\hat{Y} = \Omega_{2,1}X + \Omega_{2,2}Y + \Omega_{2,3}Z.$$

By using $[X,Y] = 2Z$ and integrating by parts, we obtain

$$(XP_tf)(0) = \int_{\mathbf{SU}(2)} \left(\Omega_{1,1}p_t + \frac{1}{2}Y(\Omega_{1,3}p_t)\right)(Xf) + \left(\Omega_{1,2}p_t - \frac{1}{2}X(\Omega_{1,3}p_t)\right)(Yf)d\mu$$

and

$$(YP_tf)(0) = \int_{\mathbf{SU}(2)} \left(\Omega_{2,1}p_t + \frac{1}{2}Y(\Omega_{2,3}p_t)\right)(Xf) + \left(\Omega_{2,2}p_t - \frac{1}{2}X(\Omega_{2,3}p_t)\right)(Yf)d\mu.$$

We easily compute

$$\Omega_{1,3} = \sin\theta \sin 2r$$

and

$$\Omega_{2,3} = -\cos\theta \sin 2r.$$

By using Hölder's inequality the expected result follows from Lemma 7.4.13. □

7.4.4 Towards optimal heat kernel bounds on $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$

The method of the complex commutation to obtain the H.Q. Li inequality on the Heisenberg group needs the optimal heat kernel estimates and in fact shows that the modified kernel $p_t(r, z+4it)$ satisfies the same upper bound away from the poles. The general idea of the method is to see the kernels as the integral of a meromorphic function in the variable y and to carefully analyse its behaviour; the difficulty coming when the critical point is getting closer and closer of the poles.

To both obtain optimal estimates and make the complex commutation method work, one should investigate the analytical properties in the y variable of the function under the integral in the heat kernel.
We do it here for the $\mathbf{SL}(2,\mathbb{R})$ group. It seems simpler to obtain optimal estimates on $\mathbf{SL}(2,\mathbb{R})$ than for the $\mathbf{SU}(2)$ group, since on $\mathbf{SU}(2)$ the function is given by a countable sum. However these analytical properties on $\mathbf{SU}(2)$ are similar to th ones on $\mathbf{SL}(2,\mathbb{R})$.
Recall that on $\mathbf{SL}(2,\mathbb{R})$, the heat kernel can be written

$$p_t(r,z) = \int_{-\infty}^{\infty} \exp\left(-\frac{f(r,z,y)}{4t}\right) V(r,y)dy$$

with

$$f(r,z,y) = \mathrm{arch}(\cosh r \cosh y)^2 - (y - iz)^2$$

and

$$V(r,y) = \frac{\mathrm{arch}(\cosh r \cosh y)}{\sqrt{\cosh^2 r \cosh^2 y - 1}}.$$

By chosing the principal determination of the squareroot function and of the logarithm function in the complex plane ($\mathbb{C}-]-\infty,0]$), we have easily that the function $\mathrm{arch}(x)$ is holomorphic on $\mathbb{C}-]-\infty,1]$. But we can do better for the function $\mathrm{arch}^2(x)$ and show in fact by using Schwarz symmetry principle it is holomorphic on $\mathbb{C}-]-\infty,-1]$. We can also note the value in -1 is defined and equals $-\pi^2$.

Now let us deal with the function $g : y \to \operatorname{arch}(\cosh r \cosh y)^2$. The thing we are interested in is to obtain an holomorphic extension of the function g from the real axis. In order to do that, we note that the function $y \to \operatorname{arch}(\cosh r \cosh y)$ defined on $\mathbb{R}_+$ admits an holomrophic extension to all $Re(y) > 0$ given by

$$\operatorname{arch}(\cosh r \cosh y) + 2ik\pi$$

where k is such that

$$Im(y) \in [-\pi, \pi] + 2k\pi$$

and so the function $g : y \to (\operatorname{arch}(\cosh r \cosh y) + 2ik\pi)^2$ with the same k is holomorphic on $Re(y) > 0$.

Now we can set $g(y) = \overline{g(\tilde{y})}$ for $Re(y) < 0$ with $\tilde{y}$ the symmetric of y with respect to the imaginary axis. Using again Schwarz symmetry principle we obtain this function is holomorphic on $Re(y) < 0 \cup J \cup Re(y) > 0$ where J is the set of all intervals of the imaginary axis where the imaginary part of g is going to zero. By the explicit formula, we see that if $k = 0$ we have only to take away the y such that $\cosh r \cosh y \in]-\infty, -1]$ and that if $k \neq 0$ we also have to take away the y such that $\cosh r \cosh y \in]-1, -1]$. Eventually, we obtain that the function $y \to \operatorname{arch}(\cosh r \cosh y)^2$ admits an holomorphic extension to

$$\begin{aligned} D_r \quad = \quad \mathbb{C} - \Bigg\{ & \bigcup_{k\in\mathbb{Z}} [i\arccos(-1/\cosh r), i(2\pi - \arccos(-1/\cosh r))] + 2ik\pi \\ & \cup \bigcup_{l\in\mathbb{Z}, l\neq 0} [-i\arccos(1/\cosh r), i\arccos(1/\cosh r))] + 2il\pi \Bigg\} \end{aligned}$$

which is given by

$$(\operatorname{arch}(\cosh r \cosh y) + 2ik\pi)^2$$

where k is such that if $Re(y) \geq 0$ then $Im(y) \in [-\pi, \pi] + 2k\pi$ and if $Re(y) \leq 0$ then $Im(y) \in [-\pi, \pi] - 2k\pi$.

In the same way we can see the function $V(r, y)$ admits an holomorphic extension to the same domain D_r. Moreover near both ends of the segments taken away, the function behaves like one over squareroot. The holomorphic extension in y is given by

$$\frac{\operatorname{arch}(\cosh r \cosh y) + 2ik\pi}{\sqrt{\cosh^2 r \cosh^2 y - 1}}$$

where k is defined in the same way as above.

Note that the domain D_r is converging (in the Hausdorff sense) when r goes to 0 to the set $\{ik\pi, k \in \mathbb{Z}, k \neq 0\}$ which are precisely the poles of the holomorphic extension of the integrand in the case of the Heisenberg group.

Bibliography

[1] Andrei Agrachev, Ugo Boscain, Jean-Paul Gauthier, and Francesco Rossi. The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups. *J. Funct. Anal.*, 256(8):2621–2655, 2009.

[2] Andrei Agrachev and Paul Lee. Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds. Arxiv Preprint, 2009.

[3] Luigi Ambrosio. Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. *Adv. Math.*, 159(1):51–67, 2001.

[4] Cécile Ané, Sébastien Blachère, Djalil Chafaï, Pierre Fougères, Ivan Gentil, Florent Malrieu, Cyril Roberto, and Grégory Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[5] Dominique Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on probability theory (Saint-Flour, 1992)*, volume 1581 of *Lecture Notes in Math.*, pages 1–114. Springer, Berlin, 1994.

[6] Dominique Bakry. On Sobolev and logarithmic Sobolev inequalities for Markov semigroups. In *New trends in stochastic analysis (Charingworth, 1994)*, pages 43–75. World Sci. Publ., River Edge, NJ, 1997.

[7] Dominique Bakry, Fabrice Baudoin, Michel Bonnefont, and Djalil Chafaï. On gradient bounds for the heat kernel on the Heisenberg group. *J. Funct. Anal.*, 255(8):1905–1938, 2008.

[8] Dominique Bakry, Fabrice Baudoin, Michel Bonnefont, and Bin Qian. Subellitpic Li-Yau estimates on three dimensional model spaces. In *Potential Theory and Stochastics in Albac*, Theta series in advanced mathematics. 2009. Albac, sepbtember 4-8, 2007.

[9] Dominique Bakry and Michel Émery. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 177–206. Springer, Berlin, 1985.

[10] Dominique Bakry and Michel Ledoux. Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. *Invent. Math.*, 123(2):259–281, 1996.

[11] Dominique Bakry and Michel Ledoux. Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator. *Duke Math. J.*, 85(1):253–270, 1996.

[12] Dominique Bakry and Michel Ledoux. A logarithmic Sobolev form of the Li-Yau parabolic inequality. *Rev. Mat. Iberoam.*, 22(2):683–702, 2006.

[13] Dominique Bakry and Zhongmin M. Qian. Harnack inequalities on a manifold with positive or negative Ricci curvature. *Rev. Mat. Iberoamericana*, 15(1):143–179, 1999.

[14] Franck Barthe and Bernard Maurey. Some remarks on isoperimetry of Gaussian type. *Ann. Inst. H. Poincaré Probab. Statist.*, 36(4):419–434, 2000.

[15] Fabrice Baudoin. *An introduction to the geometry of stochastic flows.* Imperial College Press, London, 2004.

[16] Fabrice Baudoin and Michel Bonnefont. The subelliptic heat kernel on SU(2): representations, asymptotics and gradient bounds. *Math. Z.*, 2008.

[17] Fabrice Baudoin and Nicola Garofalo. Generalized Bochner formulas and Ricci lower bounds for sub-Riemannian manifolds of rank two. arxiv preprint, 2009.

[18] Robert O. Bauer. Analysis of the horizontal Laplacian for the Hopf fibration. *Forum Math.*, 17(6):903–920, 2005.

[19] Richard Beals, Bernard Gaveau, and Peter C. Greiner. Hamilton-Jacobi theory and the heat kernel on Heisenberg groups. *J. Math. Pures Appl. (9)*, 79(7):633–689, 2000.

[20] William Beckner. A generalized Poincaré inequality for Gaussian measures. *Proc. Amer. Math. Soc.*, 105(2):397–400, 1989.

[21] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.

[22] Sergey G. Bobkov. A functional form of the isoperimetric inequality for the Gaussian measure. *J. Funct. Anal.*, 135(1):39–49, 1996.

[23] Sergey G. Bobkov. An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space. *Ann. Probab.*, 25(1):206–214, 1997.

[24] Michel Bonnefont. A discrete version of the Brunn-Minkowski inequality and its stability. *Ann. Math. Blaise Pascal*, 2009.

[25] Michel Bonnefont. The subelliptic heat kernel on SL(2,R): an integral representation and some functional inequalities. Arxiv preprint, 2009.

[26] Ugo Boscain and Francesco Rossi. Invariant Carnot-Caratheodory metrics on S^3, SO(3), SL(2), and lens spaces. *SIAM J. Control Optim.*, 47(4):1851–1878, 2008.

[27] Peter Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.

[28] Huai Dong Cao and Shing-Tung Yau. Gradient estimates, Harnack inequalities and estimates for heat kernels of the sum of squares of vector fields. *Math. Z.*, 211(3):485–504, 1992.

[29] Mireille Capitaine, Elton P. Hsu, and Michel Ledoux. Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces. *Electron. Comm. Probab.*, 2:71–81 (electronic), 1997.

[30] Djalil Chafaï. Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities. *J. Math. Kyoto Univ.*, 44(2):325–363, 2004.

[31] Sagun Chanillo and Paul C. Yang. Isoperimetric inequalities and volume comparison theorems on CR manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2009.

[32] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.

[33] Thierry Coulhon and Xuan Thinh Duong. Riesz transform and related inequalities on noncompact Riemannian manifolds. *Comm. Pure Appl. Math.*, 56(12):1728–1751, 2003.

[34] Thierry Coulhon and Laurent Saloff-Coste. Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana*, 9(2):293–314, 1993.

[35] Michael Cowling and Adam Sikora. A spectral multiplier theorem for a sublaplacian on SU(2). *Math. Z.*, 238(1):1–36, 2001.

[36] Edward B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[37] Anthony H. Dooley and Sanjiv K. Gupta. The contraction of S^{2p-1} to H^{p-1}. *Monatsh. Math.*, 128(3):237–253, 1999.

[38] Sorin Dragomir and Giuseppe Tomassini. *Differential geometry and analysis on CR manifolds*, volume 246 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2006.

[39] Bruce K. Driver and Tai Melcher. Hypoelliptic heat kernel inequalities on the Heisenberg group. *J. Funct. Anal.*, 221(2):340–365, 2005.

[40] Nathaniel Elredge. Precise estimates for the subelliptic heat kernel on H-type groups. Arxiv Preprint, 2008.

[41] Nathaniel Elredge. Gradient estimates for the subelliptic heat kernel on H-type groups. Arxiv Preprint, 2009.

[42] Alexandre Engoulatov. A universal bound on the gradient of logarithm of the heat kernel for manifolds with bounded Ricci curvature. *J. Funct. Anal.*, 238(2):518–529, 2006.

[43] Jacques Faraut and Khélifa Harzallah. *Deux cours d'analyse harmonique*, volume 69 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1987. Papers from the Tunis summer school held in Tunis, August 27–September 15, 1984.

[44] Éric Fontenas. Sur les constantes de Sobolev des variétés riemanniennes compactes et les fonctions extrémales des sphères. *Bull. Sci. Math.*, 121(2):71–96, 1997.

[45] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[46] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, 1987.

[47] Nicola Garofalo and Duy-Minh Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.*, 49(10):1081–1144, 1996.

[48] Bernard Gaveau. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. *Acta Math.*, 139(1-2):95–153, 1977.

[49] Leonard Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4):1061–1083, 1975.

[50] Leonard Gross. Logarithmic Sobolev inequalities and contractivity properties of semigroups. In *Dirichlet forms (Varenna, 1992)*, volume 1563 of *Lecture Notes in Math.*, pages 54–88. Springer, Berlin, 1993.

[51] Masanori Hino and José A. Ramírez. Small-time Gaussian behavior of symmetric diffusion semigroups. *Ann. Probab.*, 31(3):1254–1295, 2003.

[52] Lars Hörmander. On the theory of general partial differential operators. *Acta Math.*, 94:161–248, 1955.

[53] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[54] Elton P. Hsu. *Stochastic analysis on manifolds*, volume 38 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[55] Yaozhong Hu. A unified approach to several inequalities for Gaussian and diffusion measures. In *Séminaire de Probabilités, XXXIV*, volume 1729 of *Lecture Notes in Math.*, pages 329–335. Springer, Berlin, 2000.

[56] Hermann Hueber and Detlef Müller. Asymptotics for some Green kernels on the Heisenberg group and the Martin boundary. *Math. Ann.*, 283(1):97–119, 1989.

[57] Andrzej Hulanicki. The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group. *Studia Math.*, 56(2):165–173, 1976.

[58] David Jerison and John M. Lee. Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. *J. Amer. Math. Soc.*, 1(1):1–13, 1988.

[59] David Jerison and Antonio Sánchez-Calle. Subelliptic, second order differential operators. In *Complex analysis, III (College Park, Md., 1985–86)*, volume 1277 of *Lecture Notes in Math.*, pages 46–77. Springer, Berlin, 1987.

[60] Nicolas Juillet. Optimal transport and geometric analysis in Heisenberg groups. Ph-D thesis, 2008.

[61] Nicolas Juillet. Transport de mesure et courbures de Ricci synthétiques dans le groupe de Heisenberg. In *Actes du Séminaire de Théorie Spectrale et Géométrie. Vol. 25. Année 2006–2007*, volume 25 of *Sémin. Théor. Spectr. Géom.*, pages 85–104. Univ. Grenoble I, Saint, 2008.

[62] Adam Korányi and Stephen Vági. Singular integrals on homogeneous spaces and some problems of classical analysis. *Ann. Scuola Norm. Sup. Pisa (3)*, 25:575–648 (1972), 1971.

[63] Rafał Latała and Krzysztof Oleszkiewicz. Between Sobolev and Poincaré. In *Geometric aspects of functional analysis*, volume 1745 of *Lecture Notes in Math.*, pages 147–168. Springer, Berlin, 2000.

[64] Rémi Léandre. Majoration en temps petit de la densité d'une diffusion dégénérée. *Probab. Theory Related Fields*, 74(2):289–294, 1987.

[65] Rémi Léandre. Minoration en temps petit de la densité d'une diffusion dégénérée. *J. Funct. Anal.*, 74(2):399–414, 1987.

[66] Michel Ledoux. A simple analytic proof of an inequality by P. Buser. *Proc. Amer. Math. Soc.*, 121(3):951–959, 1994.

[67] Michel Ledoux. Isoperimetry and Gaussian analysis. In *Lectures on probability theory and statistics (Saint-Flour, 1994)*, volume 1648 of *Lecture Notes in Math.*, pages 165–294. Springer, Berlin, 1996.

[68] Michel Ledoux. The geometry of Markov diffusion generators. *Ann. Fac. Sci. Toulouse Math. (6)*, 9(2):305–366, 2000. Probability theory.

[69] Paul Lévy. Wiener's random function, and other Laplacian random functions. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, pages 171–187, Berkeley and Los Angeles, 1951. University of California Press.

[70] Hong-Quan Li. Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg. *J. Funct. Anal.*, 236(2):369–394, 2006.

[71] Hong-Quan Li. Estimations asymptotiques du noyau de la chaleur sur les groupes de Heisenberg. *C. R. Math. Acad. Sci. Paris*, 344(8):497–502, 2007.

[72] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.

[73] John Lott and Cédric Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[74] Patrick Maheux and Laurent Saloff-Coste. Analyse sur les boules d'un opérateur sous-elliptique. *Math. Ann.*, 303(4):713–740, 1995.

[75] Tai Melcher. Hypoelliptic heat kernel inequalities on Lie groups. *Stochastic Process. Appl.*, 118(3):368–388, 2008.

[76] John Mitchell. On Carnot-Carathéodory metrics. *J. Differential Geom.*, 21(1):35–45, 1985.

[77] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[78] Roberto Monti and Francesco Serra Cassano. Surface measures in Carnot-Carathéodory spaces. *Calc. Var. Partial Differential Equations*, 13(3):339–376, 2001.

[79] Shin-ichi Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.

[80] Fulvio Ricci. A contraction of SU(2) to the Heisenberg group. *Monatsh. Math.*, 101(3):211–225, 1986.

[81] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[82] Michel Rumin. Formes différentielles sur les variétés de contact. *J. Differential Geom.*, 39(2):281–330, 1994.

[83] Robert S. Strichartz. Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal.*, 52(1):48–79, 1983.

[84] Robert S. Strichartz. Sub-Riemannian geometry. *J. Differential Geom.*, 24(2):221–263, 1986.

[85] Karl-Theodor Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

[86] Karl-Theodor Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.

[87] Michael E. Taylor. *Noncommutative harmonic analysis*, volume 22 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1986.

[88] Michael E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1996. Basic theory.

[89] Michael E. Taylor. *Partial differential equations. II*, volume 116 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1996. Qualitative studies of linear equations.

[90] Nicolas Th. Varopoulos. Hardy-Littlewood theory for semigroups. *J. Funct. Anal.*, 63(2):240–260, 1985.

[91] Nicolas Th. Varopoulos. Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique. *Bull. Sci. Math.*, 113(3):253–277, 1989.

[92] Nicolas Th. Varopoulos, Laurent Saloff-Coste, and Thierry Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1992.

[93] Anatolii M. Vershik and Vladimir Ya. Gershkovich. Nonholonomic dynamical systems. Geometry of distributions and variational problems. In *Current problems in mathematics. Fundamental directions, Vol. 16 (Russian)*, Itogi Nauki i Tekhniki, pages 5–85, 307. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987.

[94] Max-K. von Renesse and Karl-Theodor Sturm. Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.*, 58(7):923–940, 2005.